CRITICAL STUDY OF THE BAKHSHĀLI MANUSCRIPT

BY

SUSHMA ZADOO

Thesis submitted for the Award of the Degree of Doctor of Philosophy in the Faculty of Social Sciences (Ancient Indian Mathematics)

1992

CENTRE OF CENTRAL ASIAN STUDIES
UNIVERSITY OF KASHMIR
SRINAGAR - 190 006

CRITICAL STUDY OF THE BAKHSHĀLI MANUSCRIPT

BY

SUSHMA ZADOO

Thesis submitted for the Award of the Degree of Doctor of Philosophy in the Faculty of Social Sciences (Ancient Indian Mathematics)

Supervisor : Dr. B.K. Kaul Deambi

1992

CENTRE OF CENTRAL ASIAN STUDIES
University of Kashmir
Srinagar-190 006

DECLARATION

This is to certify that the thesis entitled "Critical Study of the Bakhshālī Manuscript" by Mrs. Sushma Zadoo in fulfilment of the requirement of the Doctoral programme, is the original work, carried out by her under my supervision. It is also certified that this work has not been submitted in this University or any other University so far.

Dr. A.M. MATTOO
Director
Centre of Central Asian Studies
University of Kashmir
Srinagar, Kashmir

B.K. Deambi

Dr. B.K. KAUL, Deambi,
Reader
Centre of Central Asian Studies
University of Kashmir, Srinagar
Kashmir
SUPERVISOR

CONTENTS

PREFACE

The Bakhshāli Manuscript written on birch-bark was discovered in 1881 in a village called Bakhshāli in the Peshawar district situated near the Pak-Afgan border. The Manuscript is written in Sanskrit in Śāradā characters and has been assigned on paleographic grounds to 12th century A.D. Stated as a great discovery by the Orientalists all over the world the Manuscript contains a very valuable work on Arithmetic and Algebra making significant contribution to the science of mathematics in the early medieval period. There is also sufficient material shedding light on the everyday life of the contemporary people.

The Manuscript, the beginning and end of which are not preserved leading to the loss of the name of the work and the author, contains problems involving rule of three, rupona method or summation of series, ratio and proportion, the computation of gold, least common multiple, square-root, plan of writing equations, etc. Since the problems discussed pertain to every day life, they are of exceptional interest. The Manuscript provides some information about the social and economic conditions of the contemporary people. Of particular interest are the references to animals, foods, metals and minerals and to affairs of God and men, some of which are of exceptional interest.

The Bakhshāli Manuscript though edited with the facsmilies in 1910 by Dr. Kaye has not yet been critically studied nor has its place been evaluated in the ancient and medieval Indian mathematical literature. Besides, the social content of the work has not been critically analysed so as to get a glimpse into the life of the people of the region represented by the Manuscript. Hence the present study.

The Bakhshāli Manuscript was first evaluated by Prof. Aldoof Hoernle who presented a study of a few portions of the text in the *Indian Antiquary*, Vols. XII, XVII

(1883, 1889 respectively). It was subsequently edited by Dr. G.R. Kaye with text, transliteration, facsmilies and a comprehensive introduction. The conclusions arrived at by Hoernle and Kaye were critically examined by B.B. Datta in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXI, 1929. A general treatment of the Manuscript has been given by B.B. Datta and A.N. Singh in their *History of Hindu Mathematics;* A.K. Bag in his *Mathematics in Ancient and Medieval India* and by Brij Mohan in his 'Ganit ka Itihas'.

The valuable contributions of these distinguished scholars especially those of Aldoof Hoernle, G.R. Kaye and B.B. Datta have provided valuable guidance and help in the present study.

The entire study has been presented in eight chapters. In Chapter I titled Introduction a brief history of mathematical studies in ancient and medieval India has been attempted to provide a necessary background for the study of the mathematical contents of the Manuscript. In Chapter II an account of the discovery and a brief description of the Manuscript has been given. The Manuscript is written in Sanskrit but the rules of Grammar have not been followed strictly. Besides the vernacular influence is predominent throughout. The script used is the Śāradā alphabet which made its appearance in 9th century A.D. and was extensively used in the entire north western part of the sub-continent. In chapter III linguistic and paleographic features of the Manuscript have been critically examined. For this chapter Hoernle's analysis of the language of the Manuscript as given in the Indian Antiquary, Vol. XVII, G.R. Kaye's account of the script employed in Manuscript as given by him in the Introduction of his Edition of the text and B.K. Kaul Deambi's critical study of the Śāradā alphabet used in the Manuscript made by him in section I origin and development of the Śāradā script of his *Corpus of Śāradā Inscriptions of Kashmir* have been fully utilized.

Lot of controversy has gathered round the question of the age of the Manu-

script. Scholars have put forth different theories with regard to the date of the Manuscript. All these theories have been critically examined in chapter IV titled "The Age of the Bakhshāli Manuscript". Chapter V contains the critical analysis of the contents of the Manuscript. The scheme of exposition, fundamental operations, fractions, arithmetical notation, word numerals, symbol for the unknown quantity, use of zero in the Manuscript have been discussed in detail. Rules followed by examples and solutions of different arithmetical and algebraic problems as given in the Manuscript have been studied in this chapter.

Socio-economic life of the people of the region represented by the Manuscript forms the subject of the VIth chapter. Owing to the extremely fragmentary condition of the Manuscript, the light thrown by the preserved portion on the contemporary social life is very meagre. The different scraps of information have been pieced together to present some gleanings of the contemporary society. As far the economic life of the people, the information furnished by the Manuscript regarding weights and measures, currency, money measures, sources of revenue, occupations of people etc. has been discussed in detail. The weights and measures as given in the Manuscript have been compared with those given by other celebrated Indian Mathematicians of ancient and medieval India. The tables assiduously prepared by Kaye have been fully utilized for this purpose. Some information contained in the Manuscript about the administrative set up and the composition of the army has also been discussed.

Some mathematical problems included in the text contain allusions to the religious beliefs of the people and to gods, godesses, demi-gods, epic-heroes, etc. The same have been analysed in chapter VII as to present some gleanings of the religious life of the people. The astronomical data found in the Manuscript has also been discussed.

The whole study has been concluded with the discussion on the relationship that Bakhshāli Manuscript bears with other important treatises on mathematics composed by the Indian authors. Kaye has traced foreign influences in the text contained in the Manuscript. Kaye's view point has been discussed and analysed in the light of contrary observations made in this regard by scholars like B.B. Datta and others.

It is my proud previlege to express highest veneration and heartfelt gratitude to Dr. B.K. Kaul Deambi, for his enthusiastic interest, constant encouragement, valuable guidance and unending zeal during the course of present study. I am also highly thankful to Prof. A.M. Mattoo, Director, Centre of Central Asian Studies for his encouragement. I wish to record heartfelt thanks to all the members of faculty of the Centre of Central Asian Studies.

I am deeply indebted to Dr. C.B. Pandey, Editor, Indira Gandhi National Centre for Arts; Dr. Advaitvadini Kaul, Assistant Editor and to the Library staff of IGNCA for their inate courtesy and valuable help.

My thanks are due to the concerned people of the libraries of Research and Publication Department (J & K); Centre of Central Asian Studies, Iqbal Library of Kashmir University; Sri Pratap Singh Library, Archeological Survey of India, Srinagar; Central Archeological Library, New Delhi; National Library, Calcutta for their cooperation in providing me all necessary Library facilities.

Last but not the least I record my sense of gratitude to the members of my family with whose help and encouragement the present study could be completed.

Sushma Zadoo

Date :

LIST OF ABBREVIATIONS

ASI	Archeological Survey of India.
BCMS	Bulletin of the Calcutta Mathematical Society.
BMA	Bulletin of the Mathematical Association
EI	Epigraphia Indica
IA	Indian Antiquary
IHQ	Indian Historical Quarterly
JA	Journal Asiatique
JASB	Journal of Asiatic Society of Bengal
JNSI	Journal of Numismatic Society of India

CHAPTER I

INTRODUCTION

A BRIEF HISTORY OF MATHEMATICAL STUDIES IN ANCIENT AND EARLY MEDIEVAL INDIA

I

MATHEMATICAL STUDIES IN ANCIENT INDIA

India that has the honour of being one of the oldest civilisations of the world has a proud past in the field of mathematics, too. The earliest traces are preserved in the 5000 year old ruins of a city at Mohenjo-Daro and allied sites. Evidence of wide streets, brick dwellings and apartment houses with tiled bathrooms, covered city drains and community swimming pools, indicate a civilization as advanced as that found anywhere else in the ancient orient. These early people had systems of writing, counting, weighing, measuring and they dug canals for irrigation. All this required considerable knowledge of basic mathematics and engineering. The people who laid the foundation of the subsequent civilization in India called the Vedic civilization were no less conversant with the science of mathematics and the allied disciplines.

Gaṇita which literally means the science of calculation is the ancient Indian name for mathematics. The term is a very ancient one and occurs copiously in Vedic literature. The *Vedāṅga-Jyotiṣa* gives it the highest place of honour among the sciences which form the *Vedāṅga* : "As the crests on the heads of peacocks, as the gems on the hoods of the snakes, so is the *Gaṇita* at the top of the Sciences known as Vedāṅga."[1] According to *Vedāṅga Jyotiṣa, Gaṇita* was then same as *Jyotiṣa* and *Jyotiṣa* has been defined as *Kāla Vijñāna-Śāstra*, which means 'the science of calculation of time'[2]

The study of mathematics as a well developed discipline starts with the

sūtra period. Since sacrifice was the prime religious avocation of the Vedic people, the knowledge of astronomy and geometry was chiefly needed for the proper construction of the sacrificial altars and for fixing the proper time for the sacrifice. The *Kalpa-sūtra* besides other matters of ritual gives the rules for the proper construction of sacrificial altars. It was perhaps in this connection that the study of problems of geometry, as also arithmetic and algebra, began in ancient India. Geometry was the science of altar-construction. The *Śulba-sūtra* (literally 'the rules of cord' meaning the measuring tape) which embodies the knowledge of geometry that the Vedic Hindus had, forms a part of *Kalpa-sūtra*. Thus, we find that in Vedic India, astronomy on one hand and geometry on the other were being cultivated under different circumstances, by different classes of priests having different duties apportioned to them. Arithmetic (including algebra) of course formed necessary adjunct of each. Hence the science of the *Gaṇita* in its early stage included astronomy, arithmetic and algebra.

The *Sūtras* apply to and cover each and every branch of mathematics, including arithmetic, algebra, geometry, trigonometry, conics, astronomy, calculus. In fact, there is no part of mathematics, which is beyond their jurisdiction. The critical mathematical knowledge of *Sūtra* literature led us to assume that there were mathematical works of even earlier age but they are lost. At present we know, however, of only seven *Śulba-sūtras;* those belonging to the *Śrauta-sūtra* of Baudhāyana, Āpastamba, Kātyāyana, Mānava, Maitrāyana, Varāha and Vādhula. According to B.B. Datta[3], there is a refrence to two other works, viz., *Māśaka śulba* and *Hiraṇyākesi śulba in the commentary of Karavindaswāmi on* Āpastamba-śulba. But both of them are not available now.

Baudhāyana Śulbasūtra, named after Baudhāyana, who is the oldest śulbakār, believed to have lived sometime between 800-500B.C., is the earliest and the largest in the śulba works.[4] It comprises 525 sūtras and is divided into three chapters. The first chapter contains 116 sūtras which give geometrical prepositions necessary for the construction of the sacrificial altars and deal briefly with the relative positions and spatial magnitudes of various vedīs (or 'altars'). The second chapter consists of 86 sutras of which the major portion is devoted to the description of the spatial relations in the different constructions of the *Agnis* (or the large 'Fire-altars' made of bricks) and the remaining portion explains the construction of two simplest *Agnis,* viz., the *Gārhapatya-citi* (or 'The House-Holder's Fire-altar'), and Chandaś-citi (or the *Agni* made, as it were, of *mantras* instead of bricks). The third chapter in 323 *sūtras,* describes mainly the construction of various types of *Agnis.* It describes the construction of as many as seventeen different kinds of *Kāmya Agnis* (or altars for the sacrifices performed with a view to attain definite objects) of rather complex nature. The work is of considerable importance owing to the various mathematical rules concerning the sacrificial altars, some of which involve irrational numbers, squaring the circle, indeterminate problems and elementary treatment of surds. Apart from this Baudhāyana has given a general enunciation of Pythagoras' theorem and an approximate value of $\sqrt{2}$ correct to five places of decimal; various methods of transformation of one figure to another, etc.[5]

Āpastamba śulba-sūtra, is named after Āpastamba who is believed to have lived in 800-500 B.C. The *śulba-sūtra* of Āpastamba comprises 223 sūtras (rules) for the construction of sacrificial altars and envisages geometrical rules and mathematical ideas of considerable importance.[6] It is among the earliest of the *Śulbas* being posterior to Baudhāyana, a much larger and also older

work.

The first section of the manual, gives the important geometrical propositions required for the construction of altars. The second section describes the relative positions of the various vedīs and their spatial magnitudes. Unlike Baudhāyana, Āpastamba here indicates briefly also the methods of their construction. They are of course the particular applications of the general geometrical theorems taught in the earlier section. The remaining sections of the *Āpastamba-Śulba-Sūtra,* deal with the construction of the *Kāmya Agnis.*

Kātyāyana-Śulba-sūtra or *kātyāyana-śulba-pariśiṣṭa* or *kātiya-śulba-pariśiṣṭa* contains altogether six *kaṇḍikas* (or small sections).[7] The *sūtra* is divided into two parts. The first part is composed in the style of the *sūtras* or aphorisms, while the second part is composed in verses. The earlier part is again subdivided into seven *Kaṇḍikas* containing altoget her 90 sūtras. It discusses the geometrical proportions, the different measures, relative positions and spatial relations for the different constructions of the *Agnis.* This manual does not treat of the construction of the *kāmya agnis.* It is because that subject has been treated in a different chapter of the *Kātyāyana-Śrauta-sūtra.* The second part comprises nearly 40 or 48 verses. it gives mainly a description of the measuring tape (*rajju*), the gnomon, the attributes of an expert altar-builder and also a few general rules for his conduct. According to B.B. Datta, the title *Kātyāyana-śulba-Pariśiṣṭa* or ('The Appendix to the śulba of Kātyāyana') was originally designed for this part and should be kept reserved to it, even now. For its is really a sort of an Appendix to the earlier part, the *Kātyāyana-śulba,* proper.[8]

Kātyāyana observes that the second part, especially the recapitulations in it, was meant to help those whose intellects are too poor to be able to fully grasp the inner meanings of the compositions in the *Sūtra* style. [9] The *śulba* of

Kātyāyana compared with the works of Baudhāyana and Āpastamba, presents some interesting features as it exhibits the whole body of geometrical knowledge required for the altar-builder in a more systematic form.

Mānava-Śulbasūtra. It is a part of the *Mānava-śrautsūtra*. The *Śulba-sūtra* of Manu is a small treatise composed in both prose and verse.[10] It is divided into seven *Khandas* (or parts", "sections"). In the first section there is given a description of the measuring tape, the gnomon, measures, four methods of determining the cardinal directions and also a method of constructing a square on a given straight line.

It may be noted that we do not find in the *Āpastamba* and *Baudhāyana Śulba-Sūtras* any method of determining the cardinal directions, though it is essentially necessary for the proper construction of the sacrificial-altras to have an accurate knowledge about them. They proceed on the assumption that the cardinal directions are already known. The section II-VI treat of the relative positions, spatial magnitudes and also the methods of the construction of the different *vedīs*, e.g., the *Pākayājñiki, Māruti* and *Vāruni Vedīs* which are not included in the above mentioned manuals. The last secion of the Mānava śulba-sūtra furnishes us with some hints about the sacrificial fees.

It also describes the method of the construction of the *Suparana-citi*. This *citi* is not found in other *Śulba-sūtras*. But for the head, its spatial magnitudes are the same as those of the most primitive *citi*, the *saptavidha-sāratni-prādeśacatura-sraśyena-citi*, described by Baudhāyana and others[11].

The *Maitrāyaṇiya-śulba-sūtra* is a different recension of the *Mānava Śulba-Sūtra*. They cover almost the same ground and more than that, many passages of them are identical. But still they should not be mistaken as one and the same work, the arrangement of matter in them is not parallel. And

there are also other marks of distinction between them.[12]

As far as the title *Śulba-sūtra* is concerned, the word Sūtra means an "aphorism" a short rule. It simply describes the style of the composition of the works and has practically no reference to their subject-matter. The science itself is really called the *Śulba*. And that is in fact, the original title of the manuals. As the *Śulba* deals with the science of geometry and its application as known amongst the early Indians, we conclude that the earliest Indian name for geometry was *Śulba*.

Vedāṅga Jyotiṣa, dealing with astronomy may also be regarded as one of the sources of Vedic Mathematics. It is the earliest Indian work on astronomy available in two recensions i.e., *Ṛgveda* and *Yajurveda*. A great deal of controversy prevailed about the time of the *Vedāṅga Jyotiṣa*, but the modern scholars are unanimous in date as 200 B.C.[13]

After the temporary conquest of north-west India by Alexandar the Great in 326 B.C., the Maurya Empire was established and in time spread over entire India and parts of Central Asia. The most famous Maurya ruler was king Aśoka (272-232 B.C.), some of whose great stone pillars erected in every important city of his day, are extant even today. The earliest specimens of our present number sysmbols are preserved on these pillars.[14]

In this period several works on astronomy and mathematics were composed on the basis of knowledge of the earlier Vedic-period and there is also evidence of foreign contacts.

The importance of *gaṇita* is also given by the Jaina's.[15] Their religious literature is generally classified into four branches, called *Anuyogas*, meaning the exposition of the principles. One of them is *gaṇitānuyoga* (the exposition of principles of mathematics), the other is the knowledge of *Saṁkhyāna* (or

the science of number, which means arithmetic and astronomy) which is stated to be one of the principal accomplishments of the Jaina priest[16]. The other two are *Dharmakathānuyoga* (or the exposition of the principle of religon and *Jyotiṣa* (or astronomy). Amongst the religious works of the Jains, the following are important from the mathematical point of view _ *Sūryaprajña-pati, Sthānā ngasūtra, Bhagvatīsūtra, Tattvārthādigama-sūtra* of Umāsvāti, *Anuyogadvārasūtra, Kṣetrasamāsa, Trilokasāra,* etc.[17] There were certainly other mathematical treatises by the early Jaina scholars, which are now lost.

B.B. Datta has referred to the kusumpura school of mathematics.[18]

He says that one of the greatest Jaina metaphysicians of India, Umāsvāti (150 B.C.) first cited the existence of this school of mathematics. He resided in the city of *kusumpura* (ancient *Pataliputra)* near Patna. This school was perhaps originated long before the famous Jaina saint Bhadrabahu (300 B.C.) who lived at kusumpura. The culture of mathematics and astronomy survived in this school for many centuries, where Āryabhaṭa I perhaps took his lesson in the fifth century A.D. The knowledge of arithmetic and *Jyotiṣa* is, said to be necessary for Jaina priest. For, the Jaina priest like his counterpart of the Vedic period has to find the right time and place for the religious ceremonies. Bṛhaspati Smṛti mentions that the king must show honours to astronomers before entering the court[19].

In the Buddhist literature also *gaṇanā* or *saṁkhyāna* (arithmetic) is re-garded as the first and the noblest of the arts.[20] Three classes of *Gaṇita* have been mentioned in ancient Buddhist literature: *mudrā* (finger arithmetic, gaṇanā (mental arithmetic) and Saṁkhyāna (higher arithmetic in general). One of the earliest enumerations of these three classes occurs in the *Digha Nikāya*[21] and it is also found in the *Vinaya-Pitaka*[22]. *Divyāvadāna*[23] and

Milindapañho.[24]

After Aśoka, India underwent a series of invasions, which were finally followed by the Gupta dynasty under the rule of native Indian Emperors. The Gupta period proved to be the golden age of the Sanskrit renaissance and India became a centre of learning art and medicine. Rich cities grew up and universities were founded. The mathematics here on is not treated as a separate subject but seems subservient to astronomy. The first important astronomical work, the anonymous *Sūrya-siddhānta* (knowledge from the Sun) dates from this period, probably about the beginning of the fifth century[25]. Eighteen *Siddhāntas* were composed during this period including *Sūrya-Siddhānta*. They go after the names of their authors namely Sūrya, *Paitāmaha* Vyāsa, Vaśiṣṭha, Atri, Parāśar a, Kaśyapa, Nārada, Garaga, Marici, Manu, Aṅgira, Lomaśa, Pauliśa, Cavana, Yavana, Bhṛgu and Saunaka. Out of these only five *Siddhāntas* namely, the *Pauliśa Siddhānta, Romaka Siddhānta* (Yavana Siddhānta), *Vaśiṣṭha Siddhānta, Saurya Siddhānta* (i.e. *Surya-Siddhānta)* and the *Paitāmaha Siddhānta* and a few others have survived.

Sūrya Siddhānta (c. 400 A.D.) is a standard astronomical treatise, widely accepted and followed in India, and one of the earliest of the Indian Scientific astronomical works, which began to take shape from about the fifth century A.D. at a time when old astronomical ideas and calculations came to be revised and placed on a scientific and mathematical basis.

The text in its present form is of much later development and result of many corrections and interpolations. There is a considerable agreement between the *Sūrya-siddhānta* as described by Varāhamihira in his *Pañcasidhāntikā*, with the modern *Sūrya-siddhānta*. The *Sūrya-siddhānta* gives important information on trigonometry and is of particular interest from mathematical point of view.

ĀRYABHAṬA I

The earliest Indian astronomer known to us is Āryabhaṭa I, born in 476 A.D., at Pataliputra, in Bihar. Āryabhaṭa I, perhaps completed his education in the school of Kusumpura near modern Patna. He was the first astronomer, whose work on *Siddhānta-jyotiṣa* included a chapter on mathematics. His *Āryabhaṭīya* [26] or *Āryasiddhānta* (laghu) is a famous Indian work on mathematics and astronomy divided into following four broad sections, viz.,

i. Daśagitikā (the ten Gīti stanzas),

ii. *Gaṇitpāda* (Mathematics),

iii. *Kālakriyā* (Reckoning of time) and

iv. *Gola* (sphere).

The work was written in 499 A.D., when he was 23 years old.[27]

"The *Āryabhaṭiya* is essentially a systematization of the results contained in the *Siddhāntas* and it is of particular value because of the picture it gives of the state of mathematical knowledge of the period. Āryabhaṭa I gives methods of solution of simple and quadratic equations, indeterminate equations of first degree. He devotes considerable attention to trigonometry and his introduction of sines and versed sines is a notable improvement upon the clumpsy half chords of Ptolemy. He hits upon a remarkably close approximations to the ratio of the circumference of a circle to its diameter (i.e. π = 3.1416), which is undoubtedly an achievement over the mathematicians of the world. He also gives correct generalised rules for computing the sum of natural numbers and of their squares and cubes." It is in the wider range including algebra, arithmetic, geometry and trigonometry that Āryabhaṭa consideres the science of *Gaṇita*.

VARĀHMIHIRA

Varāhmihira was a native of Avanti and pupil of his astronomer father Ādityadāsa. He was a *Magadha Brāhmaṇa,* born about 505 A.D. He is said to have died in 587 A.D. and around 550 A.D. must have composed the following works. His three works are _

i. *Bṛhajjātaka*[28].

ii. *Bṛhatsaṁhita*[29] and

iii. *Pañcasiddhāntikā*[30]

Bṛhajjātaka is an astrological work and its importance lies in the use, probably for the first time in india, of the Zodiac, with Greek names of the zodiacal signs and planets. Similarly *Bṛhatsaṁhita* is important from the point of view of the history of astronomy. *Pañcasiddhāntika* is also considered important in the history of astronomy as it gives the description of *Paitāmaha Siddhānta, Romaka-Siddhānta, Pauliśa Siddhānta, Sūrya Siddhānta* and *Vaśiṣṭha Siddhānta.* In the history of mathematics also, the work has a high place for its amount of trigonometrical information. It gives different relations among three functions, *jyā, kojyā, utkramajyā* and value of different *jyās* in a quadrant, drawn at a fixed interval (sine table) besides various other information of mathematical importance.[31]

BHĀSKARA I

Bhāskara I flourished in c.600 A.D. It is said that he belonged to the part of

India near Valabhi and that he imbibed his knowledge of astronomy from his father and was undoubtedly the most competent exponent of Āryabhaṭa 1's school of astronomy.

His three works are as under _

i. *Āryabhaṭiya-sūtrabhaśya* or *Āryabhatiya-Tantrabhāśya*, a commentary on the Āryabhaṭiya of Āryabhaṭa I.

ii. *Laghubhāskarīya* – an abridged and simplified version of his own Mahābhāskarīya in 8 chapters.

iii. *Mahābhāskarīya*[32] or *Bṛhat-bhāskarīya*, an astronomnical treatise dealing with the duration of day and night, the method of testing the accuracy of a given position of planets, etc.

Bhāskara I was mainly an astronomer but made commendable progress in the solution of the indeterminate equation of the first degree, the method of whose solution is given in his *Mahābhāskariya* for use in the solution of astronomical problems.

BRAHMAGUPTA

Brahmgupta was the most prominent Hindu mathematician, who lived and worked in the astronomical center of Ujjain, Madya-Pradesh. Born in 598 A.D. Brahmgupta whose father's name was Jiṣṇu, wrote *Brāhma-sphuṭa-siddhānta* (the revised system of Brahma) at the age of thirty. The other astronomical work, *Khaṇḍkādyaka* has been written by him in 655 A.D. He lived during the reign of Śrīvyāghramukha, the greatest king of *cāpa* dynasty. Four chapters and a half out of twenty four chapters of his *Brāhma-sphuṭa-siddhānta* are devoted to the treatment of topics of mathematics.[33] Chapter twelve called

'the *Gaṇita'* or the *Paṭi-Gati*, treats of subjects belonging properly to arithmetic and geometry. This *Gaṇitādhyāya* (chapter on mathematics) deals with cyclic, triangle and quadrilateral, rules for arithmetical operations involving zero, negative numbers, quadratic equations. Chapter eighteenth called the *kuṭṭaka*, contains discussion of almost all the matters included in Bhaskara's algebra. This chapter, *Kuṭṭakādhyāya* (chapter on indeterminate equations) contains solutions of the indeterminate equation of both first and second degree. Ninteenth chapter deals with the sun-dial and shadow-problems which have also been partly treated in chapter twelfth. Twentieth chapter, entitled 'supplement to *Chanaściti'*; is devoted to that subject. Finally a part of twentyfirst chapter is devoted to the construction of the table of sines. This portion really belongs to trigonometry. In chapter ninth of his *Khaṇḍakhādyaka*,[34] Brahmgupta gave a method of obtaining from the given table of sines, the sines of intermediate angles.

LALLA

Lalla, the grandson of Śāmba and son of Bhaṭṭa Trivikrama, flourished in c. 638 A.D. or c. 768 A.D. His *Śiṣyadhivṛddhida*[35] in one thousand ślokas is fully devoted to astronomy and contains some important information on trigonometry. It is based on the Āryabhaṭiya of Āryabhaṭa. It is said that Lalla had written two other works namely, *Pāṭīgaṇita* and *Siddhāntatilaka* which was similar to the *Brāhmasphuṭa siddhānta* and contained chapters on arithmetic and algebra.

GOVINDASVĀMIN

Govindsvāmin seems to have flourished in Kerala in the first half of ninth

century A.D. He has written a commentary on the *Mahābhāskarīya* of Bhāskara I named *'Mahābhāskarīyabhāṣya.* His *Govindākṛiti,* that appears to be lost now, contained chapters on arithmetic and algebra.

MAHĀVĪRA

Mahāvīra, flourished about 850 A.D. and hailed from southern India. The fame of this Jaina mathematician rests on his brilliant work known as the *Gaṇita-sāra-saṁgraha.*[36] It is said that *Gaṇita-sāra-saṁgraha* was written by Mahāvīrācharya during the reign of Rāṣṭrakuṭa king Amogvarṣa.

The *Gaṇita-sāra-saṁgraha* (collection of the essence of Gaṇita) contains mainly topics from arithmetic and geometry. Mahāvīra has dealt with almost all the problems of his predecessors and has made the classification of the arithmetical operations simpler and given a number of examples to elucidate the rules. He shows ability to handle geometrical as well as arithmetical series and his most noteworthy work is the treatment of fractions. Mahāvīra, restricted the scope of Gaṇita to arithmetic and geometry and excluded jyotiṣa from its scope. That shows Mahāvīra considered *Gaṇita* separate from *jyotiṣa.*

ŚRĪDHARA

There is a great deal of controversy about Śrīdhara's time and place of origin. According to Bag[37] and Shukla.[38] who have discussed the subject at length have arrived at the conclusion that Śrīdhara flourished between c.850A.D. and c. 950 A.D.

Śrīdhara's *Paṭi-Gaṇita*[39] or *Paṭi-Gaṇita-Sāra* or *Gaṇita-sāra* (essence of the *Pāṭī-Gaṇita*) is more known as *Triśatika* as it contains 300 verses. This latter name is believed to have been given to it by some later mathematician, as it occurs nowhere in the original text. *Paṭi-Gaṇita* is a work on arithmetic and

mensuration and deals with multiplication, divisions, square, cube, square-root, cube-root, fraction, rule of three, areas of plane figures and eight rules for operation involving zero but excluding the division by zero. Śrīdhara for the first time gave a rule to extract roots of $ax^2 + bx = c$, which is known usually as Śrīdhara's formula.

ĀRYABHAṬA II

Āryabhaṭa II dealt with various problems of mathematical interest in his '*Mahābhāskarīya*'[40] an astronomical work in eighteen chapters. He is said to have flourished in 950 A.D. He separately mentions the names of three branches of mathematics, viz., the *pāṭī*, *kuṭṭaka* and *bīja*. Fifteenth chapter of his treatise, *Mahā-Siddhānta* contains the *Paṭi-gaṇita* and seventeenth chapter of the same deals with the *kuṭṭaka*. Āryabhaṭa II has suggested some corrections in the treatment of solution of simultaneous indeterminate equations of the first degree.

ŚRĪPATI

Śrīpati, son of Nāgadeva and grandson of Bhaṭṭa Keśava flourished in c. 999 A.D. According to Dalta[41], the distinguished mathematician, Śrīpati, lived in Kashmir.[41].

Śrīpati wrote *Gaṇitatilaka*[42], *Siddhāntaśekhara*[43] (in 1039 A.D.) and *Bījagaṇita* besides five other works on astronomy and astrology. The *Gaṇitatilaka* is devoted exclusively to arithmetic and the*Siddhāntaśekhara*, a work mainly on astronomy in twenty chapters deals with algebra in two chapters namely *Vyaktagaṇitādhyāya* and *avyaktagaṇitādhyāya*. *Bījagaṇita* is now lost.

BHĀSKARA II

Bhāskara II was born in the year 1036 of the Śāka kings i.e. 1114 A.D. and was born of a renowned Brāhmaṇa scholar and astronomer Maheśvara at a city called Bijjalabida. This report has been given by Bhāskara II, himself in one of his works named *'Siddhānta-śiromaṇi'*. Among the works of Bhāskara II,

Siddhānta-Śiromaṇi (diadem of an astronomical system),

Līlāvatī[44] (the beautiful i.e. the noble science) and *Bīja-gaṇita* (root extraction) are of importance.

He is the author of two other works namely *Vāsnābhāṣya*, his own commentary on *Siddhānta Śiromaṇi*, and *Karaṇakutūhala*, a treatise on planetary motion.

Siddhānta Śiromaṇi was written in 1150 A.D. and showed little advancement over the work of Brahmgupta. It is a well-known and widely used astronomical work, that consists of trigonometry including sine table and different relations among the three functions known as *jyā*, *kojyā* and *utkramajyā*. *Līlāvatī* is a well known work on arithmetic and geometry and the topics discussed in it are _ cipher, inversion, supposition, concurrence, disimilar operations, operation relative to squares, operation relative to multiplicator, rule of three (direct and inverse), rule of five, seven, nine and eleven terms, interest, barter, purchase and sale, allegation, permutations and combinations, progression (arithmetical and geometrical), plane figures, excavation, stocks, saw, maunds of grain, shadow of gnomon, pulverisor, combination. This work of Bhāskara was translated into persian by Fyzi in 1587 A.D. by the direction of the emperor Akbar. It is said that *Lilavati* was the name of Bhāskara II's daughter. *Bījagaṇita* is a well known algebraical work containing six fundamental operation in algebra: addition, subtraction, multiplication,

division, squaring and extraction of square-root. Bhāskara applies these six operations to positive and negative quantities, i.e. the laws of sign, zero, monomial, polynomial and surds; then he treats of :

i. *kuṭṭaka* (pulveriser), that deals with the complete general solution of the indeterminate equation of the first degree and ii. *Varga Prakṛti* (affected square) with *cakravāla* (cyclic method) that deals with the general solution in rational integers of the Pillian equations. The full solution of the quation and of its more general form — $ax^2 + bx + c = y^2$, was given by Bhāskara.[45]

The above survey does not include *Bakshāli Manuscript*, a detailed study of which will follow in the subsequent chapters.

REFERENCES

1. *"Yathā śikhā mayurāṇām nāgānām maṇayo yathā*
   Tadvadvedāṅgaśastrāṇām gaṇitaṁ mūrdhani sthitaṃ"

   Vedāṅga Jyotiṣa, 4.

2. *Ibid., Vedāṅga Jyotiṣa,* Ed. by A Weber, Berlin, 1862; Ed. with English translation by R. Shyama Sastri, Mysore, 1936; Ed. by A. Weber, Allahabad, 1960.

3. Datta, B.B. *The Science of the Śulba,* A study in early Hindu Geometry, Calcutta University, 1932, p.2

4. *Baudhāyana Śulba Sūtra,* Ed. by W. Caland, Vol. 3, Calcutta, 1913.

   cf., Datta, *The Science of the Śulba;* Bag, A.K. *Mathematics in Ancient and Medieval India,* Delhi, 1979, p.5; Sen, S.N. *A Bibliography of Sanskrit works on Astronomy and Mathematics,* Part I, New Delhi, 1966, p. 16.

5. Bag, *Mathematics in Ancient and Medieval India,* p. 5.

6. *Āpastamba-Śulba Sūtra,* Ed. by D. Srinivasanchar and V.S. Narasimachar, Mysore, Mysore Sanskrit Series, No. 73, 1931.

   cf., Bag, *op.cit.,* p.5; Sen, *op.cit.* p.5; Datta, *op.cit.,* pp. 3, 4.

7. *Kātyāyana Śulba Sūtra,* Ed. by M.M. Pathaka, Benaras, 2 Vols, 1900.

   cf., Sen, *op.cit.,* p. 107; Datta, *op.cit.,* p.4.

8. Datta, *op.cit.,* p.4.

9. *Kātyāyana Śulba Sūtra, op.cit.;* Datta, *op.cit.,* p. 4.

10. *Mānava Śulba Sūtra,* Ed. with English translation by J.M. Van Gelder, New Delhi, 1961.

    cf. Datta, *op.cit.,* p. 5; Sen, *op.cit.,* p. 142.

11. Datta, *op.cit.*, p. 6.

12. *Ibid.*

13. *Vedāṅga Jyotiṣa, op.cit.*

    cf. Bag, *op.cit.*, p. 6; Sen, *op.cit.*, p. 120.

14. Eves, Howard, *An Introduction to the History of Mathematics*, p. 7.

15. *Bhagvati-Sūtra*, Ed. by N.L. Vaidya with the commentary of Abhayadeva Suri, Srivijayadeva Sursangha Series No. 8, 1954; *Anuyogadvārasūtra*, Ed. with Hemacandra's commentary, Jaina Pustakoddhara Series No. 37, Bombay, 1916; *Triloksāra*, Ed. by N. Premi with the commentary of Mādhava Candra, Bombay, 1919.

16. *Bhagvatī-Sūtra*, sūtra 90.

17. *Sūryaprajñāpati*, Ed. with English translation and Notes by G. Thibaut, *JASB*, 49, pp. 107-27, pp. 181-206, 1880; *Sthānāṅgasūtra*, Ed. N.L. Vaidya with the commentary of Abhayadeva Suri, Srivijayadeva Sursangha Series No. 8, 1954; *Bhagvati-Sūtra, op.cit.*; *Tattvārthādhigamasūtra*, Ed. by A.S. Sastri, Govt. Oriental Library Sanskrit Series No. 84, Mysore, 1944; *Anuyogadvārasūtra, op.cit.*; *Kṣetrasamāsa* and *Trilokasāra, op.cit.*

18. Datta, B.B. The Jaina School of Mathematics, *BCMS* Vol XXI, 1929, pp. 126-128.

19. *Bṛhaspati Smṛti*, I. 20.

20. *Vinaya Piṭaka*, Ed. by Oldenberg, Vol. IV, p. 7; *Majjhima Nikāya*, Vol. I, p. 85; *Cullaniddesa*, p. 199.

21. *Digha Nikāya*, I. p. 51.

22. *Vinaya Pitaka*, op.cit., IV, p. 7.

23. *Divyāvadāna*, Ed. by E.B. Cowell and R.A. Neil, cambridge, 1886, pp. 3, 26, 88.

24. *Milindapañho*, English translation by Rhys Davids, Oxford, 1890, p. 91.

25. *Sūrya Siddhānta*, Translated into English by E. Burgess, University of Calcutta, 1935; Eves, Howard, *op.cit.*, p. 180.

26. *Āryabhaṭīya*, Ed. by H. Kern, Leiden, 1874; Ed. by S.K. Pillai, Trivandrum Sanskrit Series, 185, 1957.

    cf. Bag. *op.cit.*, pp. 15-16; Datta, B.B. The Scope and Development of Hindu Ganita, IHQ, Vol. 5, p. 487.

27. *"Ṣaṣṭhyabdānāṃ ṣaṣṭhir yadā vyatītāstrayaśca yuga pādāḥ |*

    tryadhikā viṃśatirabdāstadeha mama janmano' tītāḥ ||"

    Āryabhaṭīya, kālakriyā V. 10.

28. *Bṛhajjātaka*, Ed. by Sitaram Jha, Banaras, 1934; English translation by V. Subramanya Sastri, Mysore, 1929.

29. *Bṛhatsaṃhitā*, by Sudhakara Dvivedi, 2 Volumes, Banaras, 895-97; English translation by V. Subrahmanya Sastri and M. Ramakrishna Bhat, 2 volumes, Bangalore, 1947.

30. *Pañcasiddhāntika*, Ed. by Thibaut and Sudhakara Dvivedi, Varanasi, 1889; Reprint, Lahore, 1930.

31. Bag, *op.cit.*, p. 17.

32. *Mahābhāskarīya*, Ed. with English translation and notes by Kripa Shankar Shukla, Lucknow University, 1960.

33. Brāhma-Sphuṭa-siddhānta, Ed. with commentary of Sudhakara Dvivedi, Benaras, 1902; Ed. by Ram Swarup Sharma and Board of editors, 4 vols., Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.

34. *Khaṇḍakhādyaka-Karana*, English translation by P.C. Sengupta, University

of Calcutta, 1934.

35. *Śiṣyadhivṛddhida*, Ed. by Sudhakara Dvivedi, Benaras, 1886.

36. *Gaṇitasārasaṁgraha*, Ed. and translated into English with notes by T. Rangacharya, Madras, 1912.

37. Bag, *op.cit.*, p. 24.

38. Shukla, vide his edition of Patiganita of Sridhara, Introduction, p. XXXV-XLIII.

39. *Paṭi-Gaṇita*, Ed. with English translation by Kripa Shankar Shukla, Lucknow University, 1959.

40. *Mahābhāskarīya*, Ed. with his own commentary by Sudhakara Dvivedi, Benaras Sanskrit Series No. 36, Benaras, 1910.

41. Datta, B.B. and Singh, A.N. *History of Hindu Mathamatics*, A Source Book, Part I and II, Single Volume Edition, Bombay, 1962, part I, p. 128.

42. *Gaṇita-tilaka*, Ed. with introduction by H.R. Kapadia, Gaekward Oriental Series No. 78 Baroda, 1937.

43. *Siddhānta śekhara*, Ed. by Babuaji Misra, 2 volumes, Calcutta University, 1936 and 1947.

44. *Līlāvatī*, Translated into English by H.T. Colebrooke vide his *'Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmgupta and Bhaskara*, London, 1817; republished by Haran Chandra Banerjee, Allahabad, 1967.

45. cf. Datta, The Scope and Development of Hindu Ganita, *IHQ*, Vol. 5, Reprint, 1985, Delhi, pp. 507, 510; Bag, *op.cit.*, pp. 26-28; Howard Eves, *op.cit.*, p. 181; Sen *op.cit.*, pp. 20, 27.

CHAPTER II

DISCOVERY, PROVENANCE AND DESCRIPTION OF THE BAKHSHALI MANUSCRIPT

II

DISCOVERY, PROVENANCE AND DESCRIPTION OF THE BAKHSHĀLI MANUSCRIPT

The Manuscript called the *Bakhshāli Manuscript* after the name of the site of its discovery was found in May, 1881 in the village of Yusufzai sub-division of the Peshawar district of the North-west frontier province of Indian sub-continent now under Pakistan. The village is about 234 km from Kabul, 249 km from Srinagar, 78 km from Peshwar, 546 km from Balkh and 109 km from Taxila ancient Taksasila.

In ancient times it formed part of Gandhāra which included the districts of Peshawar and Rawalpindi. Owing to its geographical position, Gandhāra was a meeting ground of the cross currents of different cultures, ethinic groups, languages and literatures. Gandhāra is mentioned as Gadar in the Bahistun Inscription of Darius.[1] It was one of the twenty three strapies forming part of Achaemenian Empire. Gandhāra is also mentioned in the Ṛgveda where good wool of the sheep of Gandhārites is referred to[2]. In the middle of the sixth century B.C. it was ruled by king Pukussati, who was the contemporary of the king Bimbasāra of Magadha[3]. The Gandhāra region remained subjected to diverse influences of both east and west e.g., Persian, Greek, Roman, Central-Asian, Indian, etc., because of its geographical location.

Regarding the discovery of the *Bakhshāli Manuscript*, a letter dated the 5th July, 1881 from the Assistant Commissioner of Mardan, has been reproduced by Dr. G.R. Kaye[4]. This letter does not appear to be wholly reliable as accepted even by Dr. Kaye as the discription contained in the letter pertained to a papyrus manuscript and not to our Manuscript which is written on birch-bark.

According to Alexander Cunningham, the Manuscript was found in a field near a well outside the village Bakhshali which is situated on the top of a mound.[5] There was no trace of any building near the spot and the Assistant Commissioner's story that it was found by a labourer while digging in a ruined stone enclousure does not appear reliable. The Manuscript soon after its discovery was sent to the Lieutenant Governor of Punjab who on the advice of Alexander Cunningham directed it to be transmitted to Dr. A.F.R. Hoernle, then head of Calcutta Madarsa for examination and publication. A short description of the Manuscript was given by Dr. Hoernle in 1882 before the Asiatic Society of Bengal and this description was published in the *Indian Antiquary* of 1883. A fuller account of the Manuscript was published in the proceedings of seventh Oriental Conference held at Vienna. An account of the Manuscript with some additions appeared also in the *Indian Antiquary* of 1888.

In 1902, the Manuscript was presented by Dr. Hoernle to the Bodlian Library of Oxford University, where it is now preserved. It was later edited by Dr. G.R. Kaye in 1912. Complete description of the Manuscript has been given by Dr. Kaye and we give below only the important features of the Manuscript.

DESCRIPTION

The Manuscript is written in the Śāradā characters and on leaves of birch-bark which from age have become dry, tender and extremely fragile through the careless handling of the finder. The Manuscript is in the mutilated condition, both with regard to the size and the number of the leaves. The present size of the leaf (see plate) is about 15 cm by 8 cm. The original size must have been about 17.5 cm by 20.625 cm. The truth about the size of the Manuscript has been confirmed by Dr. Hoernle himself on the well known fact that the old birch-

bark manuscripts were always written on the leaves of a squarish size.

Since the beginning and the end of the Manuscript are lost, both the title of the Manuscript and the name of its author are unknown. The extant Manuscript consists of 70 birch-bark leaves. Out of these 70 leaves some leaves are mere scraps. The largest leaf of the Manuscript measures 14.5 cm by 8.9 cm. Thirty five leaves of the Manuscript are in fair condition but broken at edges.

The size of these thirty five leaves is not less than 12.5 cm by 7.5 cm. Sixteen leaves are in fair condition but more damaged than earlier 35 leaves. The size of these sixteen leaves is not less than 11.8 cm by 5 cm. Again, seven leaves are more damaged than these leaves. Now out of left over 12 leaves, one leaf is entirely blank & eleven leaves are mere scraps.

Dr. Hoernle, the first scholar who worked on the Manuscript says that the *Bakhshāli Manuscript* seemed to be a large work, perhaps divided into chapters or sections. The existing leaves include only middle portion of the work or of a division of it.[6] The preserved text contains portions between *sutra* 9 to *sutra* 57 only.

The Bakhshāli Manuscript contains a very valuable work on Mathematics making significant contribution to the study of the science of Mathematics in ancient India. The Manuscript deals with both Arithmetic or Algebra involving systems of linear equations, rule of three, indeterminate equations, arithmetical progressions, square-root, profit and loss, time and distance and problems on income and expenditure, the computation of fineness of gold etc. Detailed discussion on the contents of the Manuscript will follow in the subsequent chapters.

REFERENCES

1. Sen, Sukumar, *Old Persian Inscriptions*, Calcutta, 1941, p. 6.
2. *Rgveda* I. 126. 7.
3. Davids, T.W. Rhys, *Buddhist India*, London, 1903, p. 28.
4. Kaye, G.R. *The Bakhshāli Manuscript*, Archaeological Survey of India, New Imperial Series, Vol. XVIII, Calcutta, 1927, reprinted, Indian Historical Researches, Vol. 24(i) p. 1.
5. *Ibid*, p. 1.
6. Hoernle, A.R.F., The Bakhshāli Manuscript, *Indian Antiquary*, Vol. XVII, 1889, p. 33.

CHAPTER III

THE LANGUAGE AND THE SCRIPT OF THE MANUSCRIPT

III

THE LANGUAGE OF THE BAKHSHALI MANUSCRIPT

According to Hoernle* the Bakhshali text is written in the so-called Gāthā dialect or in that literary form of the north-western Prākṛit which preceeded the employment in secular composition of the classical Sanskrit. Its literary form consisted in what may be called (from the Sanskrit point of view) an imperfect Sanskritisation of the vernacular *Prākṛit*. This form of Sanskrit language appears to have been in general use, in north-western India, for literary purposes till about the end of the 3rd century A.D. Hence it exhibits at every turn the peculiar characteristics of the underlying vernacular." The language of the Manuscript appears to be what has been called the hybrid Sanskrit i.e. Sanskrit full of vernacular influences, the same type that we find in the Buddhist texts in Sanskrit, discovered from this part of the country.

The following peculiarities of the language deserve mention—

1. INSERTION OF EUPHONIC CONSONANTS: of *m, in eka-m-ekatvaṁ, bhṛitako-m-ekapaṇḍitaḥ;* of *r* in *tṛi-r-aśīti, labhate-r-ashṭou.*

2. INSERTION OF s : in *vibhaktaṁ-s-uttare,* kshīyate-s-traya.

3. DOUBLING OF CONSONANTS : in compounds, *Prathama-d-dhānte, eka-s-saṁkhyā;* in sentences : *yadi-sh-shaḍbhi,* ete-s-samadanā.

4. PECULIAR SPELLINGS : *tṛiṅśā* or *tṛimśa* for *triṁśat.*The spelling with the guttural nasal before *ś* occurs only in this word; not otherwise, e.g. *chatvālimśa* 40. Again *ṛi* for *ri* in *tṛidine, kṛiyate, vimiśṛitam, kṛiṇāti;* and *ri* for *ṛi* in *riṇam, drishṭaḥ.* Again *katthyatām* for *kathyatāṁ*. Again the *jihvāmūlīya* and the *upadhmānīya* are always used before gutturals and palatals respectively.

5. IRREGULAR SANDHI : *Ko, so, rā* for *kaḥ, sa, ra;* dvayo *kechi* for *dvayaḥ kechi, dvayo cha* for *dvayaś cha, dvibhi kri* for *dvibhiḥ kri, adyo vi* for *adyor vi,* vivaritāsti for *vivaritam asti.*

6. CONFUSION OF SIBILANTS : *s* for *sh* , in *sashṭi 60, masko;* sh for *s* , in *daśamsha, visho dhayet, sheṣ aṁ.*

7. CONFUSION OF *ṇ* AND *n* : *utpaṇṇa* for *utpanna, kṣ ayena* for kṣ ayeṇa.

8. OMMISSION OF A FINAL CONSONANT : *bhājaye, kechi,* for *bhājayet, kechit.*

9. INTERPOLATION OF r : *hriṇaṁ* for *hinaṁ.* The following are specimens of etymological and syntactical peculiarities.

10. ABSENCE OF INFLECT ION : nominative, singular, masculine; *eṣa sā rāśi* for *rāśiḥ, gavāṁ viśeṣ a kartavyaṁ* for *viśeṣ aḥ;* accusative plural, *dināra* dattavan for dinārān.

11. PECULIAR INFLECTION : gender singular, *gatisya* for *gateḥ; parasmaipad* for *atmnipad, vikṛīṇāti* for *vikriṇīte* (he sells); *atmnipad* for *parasmaipad, arjayate* for *arjayati* (he earns).

12. CHANGE OF GENDER : masculine for neutral, *mūla* for *mūlāni;* neutral for masculine, *vargaṁ* for *vargaḥ;* neutral for feminine, *yutiṁ cha kartavyā* for *yutiś.*

13. EXHANGE OF NUMBERS : plural for singular, (bhavet) *lābhāḥ* for *lābhaḥ.*

14. EXCHANGE OF CASES : accusative for nominative, dvitīyaṁ, *arjayate budhaḥ* for dvitiyaḥ, accusative for instrumental, *kṣ ayaṁ samguṇya* for kṣ ayeṇa; accusative for locative, kiṁ kālam for *kasmin kāle;* instrumental for locative, *anena kālena* for *asmin kāle;* instrumental for nominative *prathamena dattavan* for *prathamo*, or *ekena yuti* for *eko;* locative for instrumental, *prathame dattā* for *Prathamena.*

15. PECULIAR FORMS : *nivarita* for *nivṛita*, *rāja* for *ārjana*, *divaddha* for *dvardha* (one and one-half), *chatvā-limśa* for chatvārimśa (forty), *paṁchā-śama* for *pañchaśat* (fiftieth), *chau-paṁchāśam* for *chaupaṁchāśat* (fifty fourth), *chaturāśīti* for *chaturaśīti* (Eighty four), *tṛi-r-āśīti* for *triraṣīti* (Eight three), *Piṇyase* for *apinyaset*, *bhājayeta* (let it be divided) for *bhajyeta*.

Some of the pecularities we also find in the Gilgit manuscripts containing mostly the Buddhist texts.[2]

1. IRREGULAR SANDHI : *Cadraprabho eṣa* for *prabha; yatrime* for *yatreme; tathaiva* for *tathaiva*.
2. IRREGULAR INFLEXIONS : *pūja* for *pūjāṃ; buddhimantān* for *buddhimata; mi* or *mayi* for *mayā; buddhāna* for *buddhānāṃ; dharmāṇa* for *dharmāṇāṃ; sarveṣā* for *sarveṣāṃ; kāli* for *kāle*.
3. INDISCRIMINATE USE OF GENDERS : like *ayaṃ* for *iyaṃ; anyān* for *anyā; ānuśaṃsān* for *ānuśaṃsā; kusumān* for *kusumāni; ime* for *imāni*.
4. SINGULAR FOR PLURAL AND VICE-VERSA : as *janenti* for *janeti; asti sattvā* for *santi*.
5. Elision of final consonants like *tasmā* for *tasmāt*.

SCRIPT

The Bakhshāli Manuscript is written in bold and clear Śāradā characters but the writing is not uniform throughout and appears to be the work of different scribes. The Śāradā alphabet is direct descendant of the Brāhmi alphabet of north western India of the sixth, seventh and the eighth centuries as found among others in the

Nirmand (district Kāngra) plate of Mahāsāmanta Mahārāja Samudrasena[3], the Hatun (Gilgit) rock inscription of Patoladeva Śāhī[4], the Gilgit manuscripts[5], coins of Pravarsena, Tormāna and of the Kārkoṭa rulers of Kashmir[6] and Bharamaor and Chattrahi inscriptions of Meruvarman (Chamba, H.P.)[7]. The earliest known records in which the Śāradā characters appear for the first time are the coins of the Utpala dynasty of Kashmir (9th century) and a brief record incised on the fragment of a broken jar discovered from the pricincts of the Avantisvāmi temple and containing the name of Avantivarma (855-883 A.D.), the founder of the temple[8]. Of about the same date is Sarāhan *Praśasti* of queen Somaprabhā, spouse of Sātyaki, a ruling chieftain of Sarāhan opposite Saho in ancient Chamba.[9] Among the other records of slightly later date, mention may be made of the Dewai (Pakistan) inscription of the Shāhi king Bimdeva (10th century)[10], the Srinagar (now Lahore Museum), and the S.P.S. Museum inscriptions of the reign of queen Diddha (A.D. 980/1 – 1003)[11], the Bharamour and Sungal (District Chamba) copper plate inscriptions of king Yugākara-varman and his son Vidagdhadeva[12], Barikot and Hund (Pakistan) inscriptions of the Shāhi king Jayapāla[13] and a few other inscriptions from Hund including that of the queen Kameśvarīdevī[14].

Śāradā is the alphabet of Kashmir par excellence and remained in use for several centuries in an extensive area of north western India including Gandhāra or north western Pakistan, Ladakh, Jammu, Himachal Pradesh, Panjab and Delhi. The alphabet continued to be used in Himachal Pradesh and Panjab upto the 13th century when it was replaced by its descendant the Devāśeṣa which in turn gave rise to the modern alphabets of Gurumukhi and Tākari. In Kashmir, however, its use continues to this day though it is confined to the older generation of the priesty class.

Considering the extend of the region over which the Śāradā alphabet remained in use for a long time, the number of Śāradā epigraphic records discovered so far is by no means very large. In all 98 inscriptions have been discovered so far,

13 in north western Pakistan, 34 in Kashmir, 6 in Jammu, 5 in Ladakh, 36 in Chamba, 3 in Kangra and one in Delhi.

On the basis of the Śāradā characters used in these records, three successive stages of development of the Śāradā alphabet can easily discerned. The earliest phase is represented by the inscriptions and coins of 9th, 10th centuries, the second by those of the 11th-13th centuries and the third and final by the epigraphic and literary records of the 14th and the sbsequent centuries.[15]

The Bakshāli Manuscript is the earliest known manuscript in which the Śāradā script has been used.

The ancient Indian Scripts and inscriptions have been a subject of study of scholars now for more than a hundred years and many a problem connected with the Indian Palaeography and epigraphy has been unravelled. But it is rather unfortunate that no systematic effort has yet been made to explain in detail the origin and the development of the Śāradā script. George Buhler in his monumental work *Indian Palaeography* has devoted one section (pp. 76-77) to the Śāradā alphabet but the veryscope of his work and the lack of material has obliged the learned scholar to treat the subject in a very brief and sketchy manner. Leeche's *Grammer of the Cashmeere Language* (Journal of Asiatic society of Bengal, 1894, pp. 399ff.) also does not give any details of this alphabet. Sir George Grierson's paper in the Journal of the Royal Asiatic Society (1916, pp. 677 ff.) merely contains the tables of ligatures of modern Śāradā and in his note in the *Linguistic Survey of India* (Vol. VIII, p. 254) he simply states that the Śāradā alphabet is the indegenous character of Kashmir and that it is generally used by Hindus and is tought in their schools in that country. The most valuable contribution is that of Dr. J.Ph. Vogel who has discussed the development of the Śāradā script at some length in his famous work "*Antiquities of Chambā State*, Part I, which must remain for ever an indespensable work of reference to a student of the Śāradā alphabet. However, vogels'

researches on the subject too, cannot be said to be complete for his treatment is confined only to the inscriptions of Chamba. Gauri Shankar Hira Chand Ojha's brief treatment of the subject in his *Bhartiya Prachina Lipimālā* is largely based on Vogel's work. Since the publicationof Vogel's work in 1911 a good number of Śāradā inscriptions came to light in north-western Pakistan and Kashmir which necessiated a revised and a complete treatment of the subject. The desideratum was fulfilled by Dr. B.K. Koul Deambi, who presented a very comprehensive and detailed scientific study of the Śāradā script in section I (Origin and Development of Śāradā script) of his *'Corpus of Śāradā Inscriptions of Kashmir,'* published in 1981.

The script employed in the Bakhshāli Manuscript has been discussed in detail by Dr. Deambi and our study of the Śāradā characters used in the Bakhshāli Manuscript will be largely based on this study. The treatment of the script of the Bakhshāli Manuscript and the conclusions arrived at by Kaye have also been critically examined in the aforesaid study.[16]

We discuss below the characters employed in the Bakhshāli Manuscript with special reference to the peculiarities which have bearing on the age of the Manuscript. The forms of the letters have been illustrated in the accompanying tables.

VOWELS

1. The initial *a* shows regularly *a* wedge at the foot of the vertical and displays two distinct forms, one, with the open top and other with the top closed.[17] The former occurs more regularly and the latter only occasionally. Kaye's assertion that *a* occurs only with an open top is not correct. The former occurs regularly in the early Śāradā inscriptions but the latter does not appear before the 11th century A.D. It is found for the first time in some coins of Mahmud of Ghazni and in the Thundu Copper plate inscription of Āsaṭa dated 1075 A.D. where, however, it

has been used only once. The earliest extant inscription in which the sign has been used with most regularity is the Khonamuh stone inscription of Zain-ul-Abidin dated 1428 A.D.[18]

2. The initial *ā* is formed like *a* with a curve open to the left and placed at the foot of the right hand vertical. The same form is consistently used in all the Śāradā records before the 15th century from which period this letter also like initial *a* occurs regularly with a closed top. The curve which marks the length of the vowel is already regularly found in the records of the 6th, 7th and the early 8th centuries A.D., e.g. in the Nirmand (H.P.) Copper Plate grant,[19] the Giligit manuscripts[20] and the Brahmor (H.P.) inscription No. 6 of Meruvarman[21]. It is also noticed in the Bower Manuscript[22] and even earlier in the 4th century A.D. in the Mathurā inscription of Candragupta II.[23]

3. The initial *i* with two dots above and the curve below found in this Manuscript presents the same form in all Śāradā records before the 16th century. Later, the two dots are joined together to form a curve.

4. The initial *ī* does not occur in our Manuscript.

5. The initial *u* has the same form as found in the records of north-western india of 4th and 5th centuries A.D. The lower curve is however elongated upwards.

6. The initial *ū* shows a developed form with a long steamer hanging down from the right top of the letter. This evidently marks the development of the earlier form found in the Sarāhan Praśasti[24] and the Chambā copper plate grant where the steamer is shorter in length and attached to the middle of the letter instead of the top.

7. Initial *ṛ* which occurs very rarely in the early Śāradā records occurs once in the word *ṛṇam* in folio 63 recto. It differs considerably from the form found in the Bower manuscript. The same form of the letter is found in the early known Śāradā

manuscripts.

8. The initial *e* shows occasionally a much developed form which is not met with in the inscriptions and which resembles the modern *e* of the Devanāgri.

9. The initial *ai, o* and *au* do not occur in the Manuscript.

Consonants

1. The letter *ka* occurs both in single and double looped forms. The form with the double loop however occurs less freqently. The latter form occurs regularly in Śāradā after the 13th century.

2. The letters *kha, ga, gha* and *ṅa* do not present anything remarkable. They occur in the same form as in the early Śāradā records. *Na* which occurs Only as a superscript letter is usually without a serif or a knob at the right top end.

3. The letter *ca* occurs mostly in quandrangular form. The archaic rounded form as found in some early Sarada records like the Sarāhan (H.P.) Praśasti (9th Century)[26], S.P.S. museum Srinagar Buddhist image inscription (10th Century)[27], Sungal (H.P.) Copper plate grant of king Vidagdhadeva of Chambā (10th-11th Century)[28] is rarely found in our Manuscript.

4. The letter *cha* which retains the archaic form does not present anything remarkable.

5. The letter *ja* occurs regularly with a dot or a knob at the right topend. The later Śāradā *ja* in which the central stroke is dropped and the top stroke replaced by two small connected curves with a small upwards stroke attached to them atthe right end is conspicuously absent in our Manuscript.

6. The letters *jha* and *ña* which occur only in legatures retain the form of the early Śāradā records.

7. The letter *ṭa* represented by a semicircle with a knob or a wedge at the right

top end does not differ from the form of the letter as found in the early Śāradā records.

8. The letter *ṭha* does not occur in our Manuscript.

9. The letter *ḍa* shows a knob in the middle and a wedge at the lower end. The development of central loop or knob is not traceable in the extant Śāradā reocrds before the 12th century A.D. It is found for the first time in the Sālhi fountain inscription dated 1170 A.D.[29] and is also found in Baijnath Praśastis dated 1204 A.D.[30]

10. The letter *ḍha* retains the archaic form.

11. The cerebral nasal occurs in the same two forms as in the inscriptions of the 11th-13th centuries. The ancient form with a small horizontal base found in the *Sarāhan Praśasti*[31] and the *Bhakund* fountain inscriptions[32] does not occur. The letter is occasionally as in *karaṇaṁ* of. (14 Recto, L.2) provided with a down stroke attached to its left end and slanting towards the right. This form of the letter resembling the form found in the Hund Stone-Inscription of queen Kamesvari Devi (11th Century)[33] and Brahmor Copper Plate grant of King Yugakra Varman of Chamba (10th-11th Century)[34] occurs regularly in the later Śāradā.

12. The letter *ta*, the most conservative letter in Śāradā is throughout uniform in shape and does not present anything worthy to note.

13. The letter *tha* is mostly lozenge in shape, the earlier crescent form occurs occasionallly.

14. The letter *da* is angular in shape and shows a wedge or a knob in the middle, as is the case with the majority of examples found in the Śāradā records. The earlier form without the thickening in the middle as found in the early Śāradā

records is rare.

15. The letter *dha* is both lozenge and circular in shape.

16. The letter *na* which resembles Devanāgri *na* does not present anything remarkable.

17. The letter *pa* with the right hand vertical protruding downwards resembles the form the letter has in the Śāradā records of the 11th and 12th centuries.

    The earlier form without the downward elongation of the right hand vertical as found in the early Śāradā records of 9th and 10th centuries does not occur. Occasionally the top of the letter is closed as in the word *Uparah* in Plate XXX, 45 Recto, l.3.

18. The letter *pha* with the lower kink turned round retains the old form.

19. The letter *ba* is mostly represented by *va* and will be described subsequently.

20. The letter *bha* with a wedge or a knob in the middle and the semi-circular curve open to the left below resembles the form of the letter as found in the Śāradā records of 11th and 12th centuries. The later form of the letter with the curve drawn up to meet the knob or wedge in the middle as found in the 14th and the subsequent centuries is found very occasionally in the text, e.g. in *bhavati*, plate XXVIII, 42 Recto, L.4. The earlier forms without the wedge or a knob in the middle as found sometimes in the Śāradā records of 9th and 10th centuries do not occur.

21. The letter *ma* also shows the Śāradā form of 11th and subsequent centuries. The earlier form without the knob in the middle is not found in the Manuscript.

22. The letter *ya* occurs both with an open top and with the top closed. We also notice a third form of the letter which shows an inward stroke issuing from the left top end. The same form, we find in the Khonmuh stone inscription of 1428

A.D.[35] It may be pointed out here that the letter occurs exclusively with close top in the later Śāradā.

23. The letter *ra* always shows a thick wedge at the lower end. This is not always the case in the inscriptions, where in place of wedge, we also find a small triangle or a small upward stroke. This feature, our Manuscript shares with all Śāradā Manuscripts.

24. The letter *la* occurs usually with two left hand curves and occasionally with the intervening horizontal stroke. This form of the letter is regularly found in the Śāradā records only after the 11th century.

25. The letter *va* which also represents *ba* occurs both in angular and cursive forms.

26. The letter *śa* occurs in a developed form with a wedge on the left and the downward prolongation of the right hand vertical. The top is represented by a horizontal bar and the earlier forms of the letter as found in the Sarāhan Praśasti (9th century) are no where found in the Manuscript.

27. The letter *ṣa* is archaic in shape and does not present anything remarkable.

28. The letter *sa* is found like *śa* with an open top. Sometimes the top of the letter is inadvertently closed and the two letters become indistinguishable.

29. The letter *ha* has the form as found in the Śāradā inscriptions of 11th and subsequent centuries.

Visarga — the *Visarga* is indicated by two usual dots placed after the consonant. The sign is also used to separate words and sentences.

The *Jihvāmūlīya* and the *Upadhmānīya* are frequently used in our manuscript. The shape of *Jihvāmūlīya* closely agrees with that of the letter *va*. The *Upadhmānīya* occurs exactly in same from as found for the first time in the copper plate inscriptions

of Chamba belonging to 11th century and later quite regularly in the inscriptions of 12th and the subsequent centuries. *Virāma* is expressed by slanting stroke running through the right top end of the vowelless consonant. The consonant with which the *Virāma* is attached appear in a very changed form. Thus *ta* becomes a mere curve and *ma* a mere dot or a small circle.

Medial Vowels—

1. The medial *ā* in our Manuscript is mostly expressed by means of a thick serief occasionally a wedge attached to the top of the consonant on the right side. In case of the consonants like *na, ta* and *na,* the a sign is expressed by a hook or a semi-circle and in case of *ja* by a small vertical attached to the right end of the central stroke. In the latter cae, the letter drops the top bar and the wedge attached to it.

   The same methods for indicating the medial *ā* are employed in the Śāradā records. However, we have an instance where the medial *ā* in *jā* has been expressed by a curve attached to the right end of the top horizontal bar. This method leading to the form of modern Śāradā is also used in the Baijnath Prashasti dated 1204 A.D.[34]

2. The medial *i* and *ī* are formed by long curves drawn to the left and right of the consonant respectively. In early Śāradā the two signs are also expressed in the archaic fashion by the use of small and skill shaped curves. The method of expressing the two medial signs in the archaic fashion gradually dropped out of use after the 12th century. In the Bakhshāli Manuscript the former method with long drawn curves has been used like the Śāradā inscriptions of 11th-13th centuries.

3. Medial *u* is expressed in two ways :

i) by a triangular wedge, which some times assumes the shape of a short upward stroke or hook, attached to the foot of the vertical on the left side. In case of consonants, like *ṇ, ḍ, ph, y* and *h* where the vertical is absent, the wedge is attached by means of a short vertical.

ii) by a curve which represents the initial *u*.

In case of *ru*, the vowel sign is denoted by attaching a downward steamer to the right of the letter.

4. Medial *ū* is also expressed like the medial *u* in two ways.

i) by a horizontal, sometimes wavy flaglike line, attached to the lower end of the vertical on the left side.

ii) by the subscribed sign for initial *ū*.

Mention may be made of the groups *rū* and *brū*. *rū* is formed by the matrika with a subscribed initial vowel mark. In our manuscript *rū* resembles the initial *u* but without the left up-stroke. In *brū* the vowel sign is expressed by two strokes attached to the letter on the right side, one slanting downwards and the other rising upwards. In our manuscript, *brū* has been formed in a similar shape but the lower slanting stroke has been attached to the foot of the abnormally lenthened right hand vertical.

5. Medial *ṛ* is expressed by a curve open to the right attached to the consonant at the lower end. The curve is rounded in the early Śāradā records and angular in the later Śāradā records. Buhler's assertion[35] that the angular medial *r* is a peculiar development of the Śāradā can not be upheld as in early Śāradā the medial *r* is more rounded in shape. In our manuscript, the curve of medial *r* is more angular than rounded as is the case with the Śāradā documents belonging to the 11th and the subsequent centuries.

6. The medial *e* in Śāradā is expressed either by a stroke horizontal or slanting- placed over the consonant or by the *pr̥ṣṭhamātrā*. i.e., by a wedge, knob or a small down stroke attached to the left end of the top bar. The latter is important as it constitutes an important proof of the age of the Manuscript. The practice of forming the medial *e* by the *pr̥ṣṭhamatrā* had become obsolete in Kashmir in the 15th century as is attested to by the evidence of Jonarāja who while recounting an anecdote regarding a forgery in a deed of sale *(vikrayapattraka)* remarks, "in order to express the *e* following a consonant the clerks formerly used to write a stroke behind the consonants. But as in course of time the script (lipi) became changed, the writers of today write the stroke expressing *e* over the consonant" The above evidence of Jonarāja is corroborated by the rare use of *pr̥ṣṭha mātrā* in forming the medial *e* in the later Śāradā manuscripts where the *pr̥ṣṭhamātrā*, as will be subsequently shown is restored to only occasionally to denote the medial *ai*. In the Bakhshāli Manuscript, both the methods have been employed. As per the calculations of Kaye, the former method with a superscribed slanting stroke has been used in 56% of the cases and the latter with the *pr̥ṣṭhamātrā* in the 44% cases. In later Śāradā (14th and subsequent centuries) the superscribed stroke is always horizontal. In our text the stroke is mostly slanting with the upper end thicker than the lower end.

7. Medial *ai* in Śāradā is expressed by the combination of two *e* symbols, i.e. by the superscribed stroke and the pr̥ṣṭhamātrā. In our Manuscript, besides the two methods, a third method has also been employed in which the sign is expressd by two top strokes. The last method is frequently used in the Śāradā records of 11th-13th centuries and is exclusively used in the later Śāradā records of 14th and subsequent centuries. As in the case of *e* the superscribed strokes are usually slanting and occasionally horizontal.

8. The medial *o* is expressed as in the Bakhshāli Manuscript and in the earlier

Śāradā inscriptions of the 10th and 11th centuries in three different ways : i. by a pṛṣṭhamātrā combined with a sign for medial *a*; ii. by the superscribed *e* stroke combined with the symbol for medial *a*; iii. by the superscribed flourish. The first two methods are rarely employed in the inscriptions after the 13th century. In the Manuscripts of the 16th and the subsequent centuries also, the two methods become obsolete and it is only the third method that is frequently employed.

9. Medial *au* is rendered throughout by the combination of superscribed flourish and the *a* mark. The method of expressing the vowel by two wedges attached to both ends of top bar combined with the superscribed *e* stroke, and used in the Sarāhan praśasti (9th century), is not employed in our Manuscript.

LIGATURES

1. *Ka* retains as in other Śāradā records its ancient form without the loop when in combination with subscript vowels *u*, *ṛ* or when forming upper and the middle element of the ligature. As the final element, it retains its usual looped shape.

2. The subscript *ñ* occurs in a form absolutely distinct from the superscript *ñ*. Its shape closely resembles the figure 3 and it occurs only in combination with *j*.

3. The lingual ṭ as the second member of the ligature occurs sometimes in its normal form as in early Śāradā records and usually it assumes a distinctive shape and consists of a semi-circular curve open to the right and a slanting stroke attached to the foot of the upper consonant on the right.

4. Generally the ligatures *ṣṭ*, *ṣṭh* appear in identical forms as is the case with the Śāradā records of 11th and subsequent centuries.

5. The subscribed dental *th* differs considerably in shape from the matrika. It is expressed as in early Śāradā records by a spiral or an inward curve drawn from left to right. In the later Śāradā records, it usually consists of a curve which

from left sharply turns round and ends in a tail on the right. It thus takes shape of the Roman letter S. In our Manuscript it is the earlier form that is exclusively used. In one instance, we have a transitional form leading to the laterform of subscript *tha*.

Buhler has drawn attention to one of the peculiar features of the Śāradā according to which *ra* as the first part of the ligature is inserted into the left side of the second letter. In general, the superscript *ra* retains its full form with the vertical slightly shortened. In ligatures *rṇ, rth, rdh*, it does not preserve its distinctive shape and usually looses its bottom part. In the ligature *rv*, it is marked only by a "small excrescence on the left curve of the *v* mātrika". In the group ry, it is expressed by a short vertical to which the curve of y is attached in one continuous storke.

In ligatures where *r* forms the middle or the final element, it is rendered by an upward stroke attached to the lower end of the upper consonant on the left.

Subscript *ya*, as final member of the ligatures is rendered by a curve open to the right.

Subscript *ta* looses its upper limb.

NUMERALS

In the Śārdā inscriptions, numerals are generally used for recording the dates. In the copper plate grants of Chambā, however, numerals have also been used for denoting the amount of the donated pieces of land. In the Manuscripts numerals are generally used for recording the number of leaves or folios, chapters or cantos. In the Baskhshāli Manuscript the numerals have been used for arithmetical notation.

In the Bakhshāli Manuscript, the systems of decimal numeration has been used. We give below a brief to description of each numeral.

The figure 1 is represented by a semi-circle. It is also sometimes expressed especially in fractions by a semi-horizontal line.

The sign for figure 2 generally consists of twocurves placed one below the other and open to the right.

The figure 3 is formed like figure 2 with addition of a small tail or a curve below the second curve.

The figure 4 looks like the ligature *ṅka.*

The symbol for the figure 5 resembles the Śāradā letter *Pa* with the right hand vertical lengthened downwards and turned towards the left.

The figure 6 resembles the final *ma.*

The sign for the figure 7 closely agrees with that used in the Nāgari to denote the figure 1.

The symbol for the figure 8 maybe described as the Śāradā *ṅa* without the wedge at the right top end and with the base stroke slanting downwards.

The figure 9 has a loop on the left and a semi-circular curve or a downward stroke.

Zero is rendered by a dot. It is also occasionally used as a sort of symbol to denote an unknown quantity.

The study of the script made above would show that our Manuscript shares most of the palaeographic characteristics of the Śāradā records of 11th and 12th centuries. Thus it will not be wrong to assign the Manuscript to this period.

REFERENCES

1. Hoernle, *IA*, Vol. XVII, pp. 36-37.
2. *Gilgit Manuscripts*, Vol. II Ed. by Nalinaksha Dutt, Second Edition, Delhi-1984, Introduction, pp. V-XI.
3. Fleet, J.F. *Corpus Inscriptionum Indicarum*, Vol. III (Inscriptions of Early Gupta kings and their successors) Varanasi, 1963, pp. 286 ff., Plate XLIV.
4. *Epigraphia Indica*, Vol. XXX, pp. 226 ft. Plate No. 36.
5. Facsimilies edited by D. Raghubir and Dr. Lokesh Chandra, New Delhi, 1963; *Gilgit Manuscripts*, Edited by N. Dutt, Srinagar, 1939; Second Edition, Delhi, 1984.
6. Cunningham, A. *Coins of Mediaeval India*, pp. 25 ff. Hindu Coinage of Kashmir, Plates III & IV.
7. Vogel, J.P. *Antiquities of Chambā State*, Part-I, Calcutta, 1911, Nos. 5-8, pp. 138 ff. Plates VII-X.
8. Deambi, B.K. Kaul, *Corpus of the Śāradā Inscriptions of Kashmir*, Delhi, 1982, No. 33, pp. 133 ff. Plate 13.
9. Vogel, *op. cit.*, pp. 152 ff. Plate XV.
10. *Epigraphia Indica*, Vol. XXI, P. 298.
11. Deambi, *op cit.*, Nos. 1 & 2, pp. 97 ff. Plates 1.22.
12. Vogel, *op cit.*, pp. 159 ff. Plate XVI & pp. 164 ff. Plate XVII.
13. *Epigraphia Indica*, Vol XXII, pp. 97 ff.
14. *Ibid*, Vol. XXI, pp. 30ff.
15. Deambi, B.K. Kaul, *History and Culture of Ancient Gandhāra and Western Himalayas*, Delhi, 1985, p.2.
16. cf. *The Bakhshāli Manuscript*, Kaye, pp. 87 ff.

17. Bakh. Ms. folio 33 verso, p. 223.
18. Deambi, B.K. Kaul, *Vishveshvarānand Indological Journal*, Hoshiarpur, Vol. XVII, pp. 220-24.
19. *Corpus Inscriptionum Indicarum*, Vol. III, pp. 286 ff, Plate XLIV.
20. Facsimiles edited by Raghuvir and Dr. Lokesh Chandra.
21. Vogel, J.Ph. *Antiquities of Chambā state*, Vol. I. No. 6, 1, 1in aditya.
22. Hoernle, The Bower Manuscript, Archaeological Survey of India (New Imperial Series) Vol. XXII.
23. *Epigraphia Indica*, Vol. XXI, pp. 8 ff.
24. Vogel, J.Ph. *Op. cit.*, No. 13, pp. 152 ff. Plate XV.
25. *Ibid*, pp. 187 ff. and plate XXV.
26. *op. cit.*
27. Deambi, op. cit. pp. 97 and Plate.
28. Vogel, *op cit.* pp 164 ff. and Plate XVII.
29. Vogel, op. cit., pp. 216 ff. and Plate XXXII.
30. *Epigraphia Indica*, Vol. I, pp. 87 ff; Buhler, *Indian Paleography*, plate V, Col. 1.
31. *op. cit.*
32. Vogel, op. cit., pp. 177 ff. Plate XX.
33. Epigraphia Indica, Vol. XXII and Plate.
34. Vogel, op. cit. pp. 159 ff. and Plate 16.
35. Deambi, op .cit. , pp. 119 ff. and Plate.

TABLE 'A'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

a	
ā	
i	
ī	
u	
ū	
r	
e	
ai	
o	
au	

VOWELS

TABLE 'B'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

ka	
kha	
ga	
gha	
ṅa	
ca	
cha	
ja	
jha	
ña	

CONSONANTS
Sheet 1

TABLE 'B'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

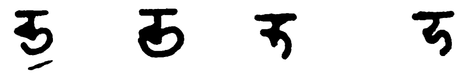
ṭa	
ṭha	
ḍa	
ḍha	
ṇa	
ta	
tha	
da	
dha	
na	

CONSONANTS

Sheet 2

TABLE 'B'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

pa	
pha	
ba	
bha	
ma	
ya	
ra	
la	
va	
śa	
ṣa	

CONSONANTS

Sheet 3

TABLE 'B'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

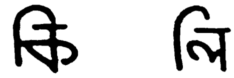
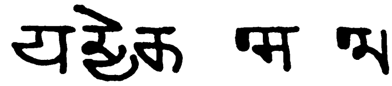
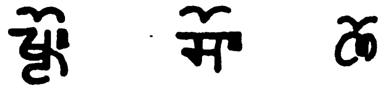
sa	
ha	
jiḥ.	
up.	
vis.	:
vir.	

CONSONANTS

Sheet 4

TABLE 'C'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

ā	
i	
ī	
u	
ū	
rū	
brū	
e	
ai	
o	
au	
hṛ	

MEDIAL VOWELS

TABLE 'D'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

kṛ	
ku	
kra	
śu	
jña	
ṣṭa	
ṣṭha	
stha	
rtha	
rdi	
rva	

LIGATURES

Sheet 1

TABLE 'D'

THE ŚĀRADĀ ALPHABET USED IN THE BAKHSHALI MANUSCRIPT

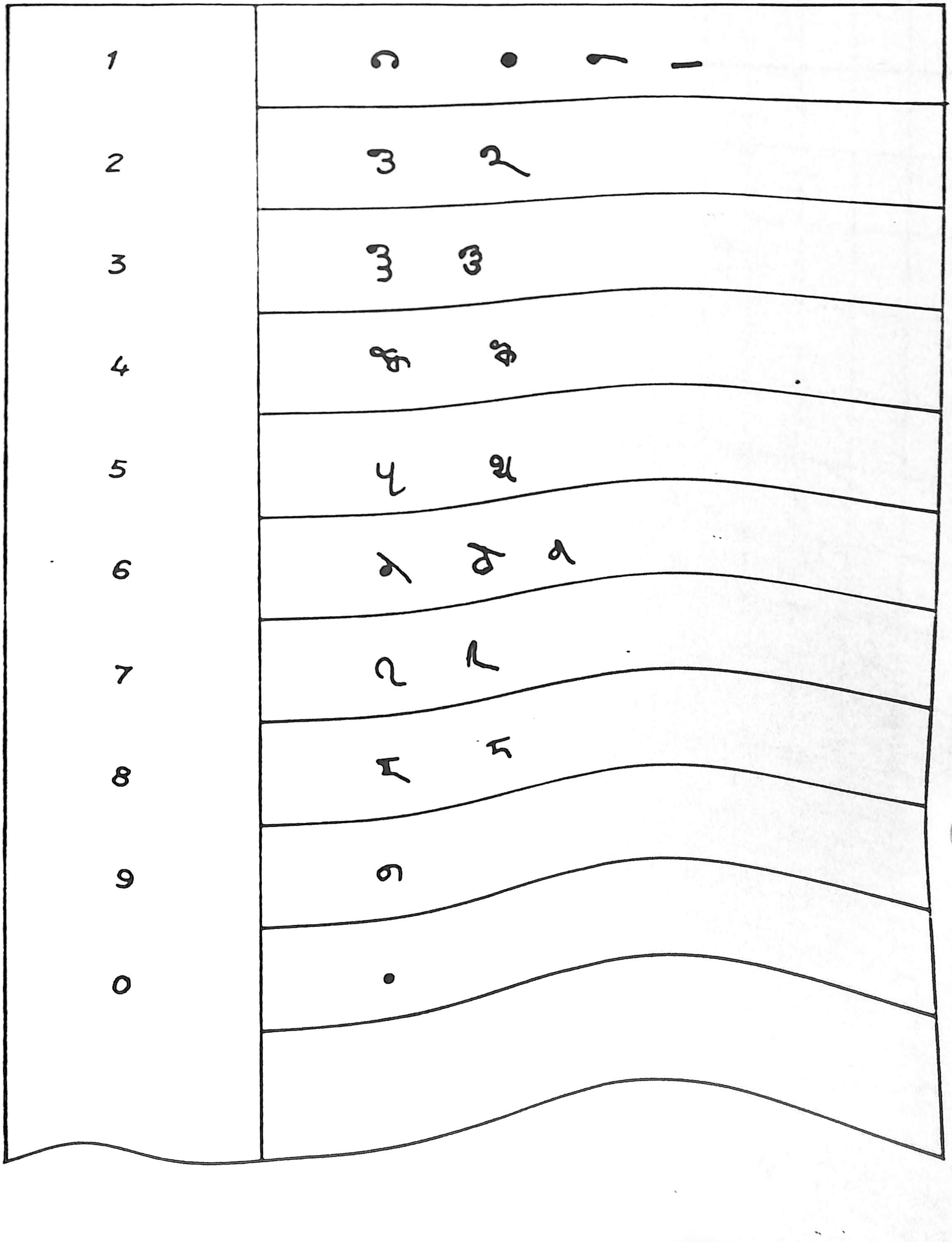
rya	
rdha	
rṇa	
pra	
krī	
trai	
ṣya	
tya	
pta	
sta	
kṣa	

LIGATURES

Sheet 2

TABLE 'E'

THE NUMERALS OF THE BAKHSHALI MANUSCRIPT



Proto Śāradā & Śāradā alphabet used in the early and Medieval Śāradā records

		Sarāhan Praśasti	Chamba Copper Plate grant	SPS. Museum Srinagar Buddhist Image Inscription	Sungal Copper plate grant of King Vidagdha of Chamba	Sālhi Fountain Inscription	Baijnāth Praśastis	Bhakund Fountain Inscription	Hund Stone Inscription of Queen Kameśvarī Devi	Coins of Mahmud of Ghazni	Thundu Copper Plate Inscription of Āsaṭa	Khonmuh Stone Inscription of Zainul-Abidin	Nirmand Copper Plate grant	Gilgit Manuscripts	Brahmaor Inscription No. 6 of Meruvarman	Bower Manuscript
a										अ अ	अ अ	अ				
ā													आ	आ	आ	आ
ū		ऊ	ऊ													
ṛ																ऋ
e		▽														
ka					क											
kha					ख											
ga					ग ग											
gha					घ											
ca		च	च		च											
ḍa						ड	ड									
ṇa		ण ण						ण	ण							
ya												य य				
śa		श														
Jihva			[illegible]													
Medial ṛ							[illegible]									
Medial au		[illegible]														

REFERENCES OF TABLES

REFERENCES OF TABLES

The detailed references of aforesaid forms taken from the Bakhshāli Manuscript are given below :

TABLE 'A'

VOWELS

1. *a*
   i. Plate XXIII, 33 *Verso*, line 6.
   ii. Plate XXXV, 52 *Recto*, L. 1
   iii. Plate II, 2 *Recto*, L. 2.
   iv. Plate XXVII, 39 *Verso*, L. 8.

2. *ā*
   i. Plate IV, 4 *Verso*, L. 7.
   ii. Plate XXXIV, 56 *Recto*, L. 4.

3. *i*
   i. Plate XXIII, 34 *Verso*, L. 4
   ii. Plate XII, 17 *Recto*, L. 2

4. *ī*
   The alphabet *ī* does not occur in our Manuscript.

5. *u*
   i. Plate II, 2 *Recto*, L. 7
   ii. Plate VI, 7 *Verso*, LL. 7,2.

6. *ū*
   i. Plate III, 3 *Verso*, L. 1.
   ii. Plate XXII, 32 *Verso*, LL. 3.
   iii. Plate XXV, 37 *Verso*, L. 2.

7. *ṛ*
   Plate XIII, 63 *Recto*, L. 5

8. *e*
   i. Plate XXIV, 36 *Recto*, L. 4.
   ii. Plate XXXIX, 58 *Recto*, L. 3.

TABLE 'B'

CONSONANTS—

1. *ka*
   i. Plate XXV, 37 *Recto*, L. 6
   ii. Plate XXV, 36 *Verso*, LL.4, 2.
   iii. Plate XXXIV, 50 *Recto*, L. 1

2. *kha*
   i. Plate XXIII, 34 *Recto*, L. 3.
   ii. Plate XXXIII, 49 *Recto*, L. 3.

3. *ga*
   i. Plate XXIII, 34 *Recto*, L. 3.
   ii. Plate XXIII, 34 *Verso*, L. 5.

4. *gha*
   i. Plate XXV, 37 *Recto*, L. 3.
   ii. Plate XII, 16 *Recto*, L. 4.

5. *ṅa*
   i. Plate XL, 60 *Recto*, L.6.
   ii. Plate XLII, 63 *Verso*, L. 5.

6. *ca*
   i. Plate V, 6 Recto, L.5.
   ii. Plate XXXII, 47 verso, L.4.

7. *cha*
   i. Plate III, 2 *Verso*, L. 6.
   ii. Plate XXVII, 39 *Verso*, L. 2.

8. *ja*
   i. Plate IV, 4 *Recto*, LL. 4, 6.
   ii. Plate XXVIII, 41 *Recto*, L. 2.

9. *jha*
   i. Plate X, 13 *Verso*, L. 4.
   ii. Plate III, 2 *Verso*, L. 6.

10. *ña*
    Plate V, 5 *Verso*, L. 1.

11. *ṭa*
    i. Plate XXII, 32 *Verso*, L. 5.
    ii. Plate XLII, 63 *Verso*, L. 5.

12. *ḍa*
    i. Plate XXXIX, 58 *Verso*, L. 3.
    ii. Plate XLV, 67 *Recto*, L. 5.

13. *ḍha*

Plate XXXIX, 58 *Verso*, L. 2.

14. *ṇa*

i. Plate XXXIX, 57 *Verso*, L.1.

ii. Plate X, 14 *Recto*, L. 2

15. *ta*

i. Plate x, 13 *Verso*, L. 5.

ii. Plate XXXIX, 58 *Recto*, L. 2.

16. *tha*

i. Plate XL, 60 *Verso*, L. 5.

ii. Plate XLI, 61 *Recto*, L. 5.

17. *da*

i. Plate II, 2 *Recto*, L. 7.

ii. Plate XLI, 61 *Verso*, LL. 5, 6

18. *dha*

i. Plate I, 29 *Recto*, L. 1.

ii. Plate II, 1 *Verso*, L. 6.

19. *na*

i. Plate XXXIV, 51 *Recto*, L. 2.

ii. Plate XXXV, 51 *Verso*, L. 5.

20. *pa*

i. Plate XXX, 45 *Recto*, L. 3.

ii. Plate XXXV, 51 *Verso*, L. 5.

21. *pha*

1. Plate XXXII, 47 *Verso*, L. 5.

ii. Plate XLV, 67 *Recto*, L.3.

22. *ba*

i. Plate XXIII, 34 *Recto*, L. 3.

ii. Plate XXXIV, 50 *Recto*, L. 3.

23. *bha*

i. Plate XXVIII, 42 *Recto*, L. 4

ii. Plate XXXVII, 55 *Recto*,
LL. 4, 5, 6, 7.

24. *ma*

i. PlateXII, 16 Verso, L.4.

ii. Plate XXXV, 52 Recto, L.3.

iii. Plate XIII, 18 *Recto*, L.4.

25. *ya*

i. Plate XXXIV, 50 *Recto*, L. 1

ii. Plate XXXL, 51 *Verso*, L. 1

iii. Plate XLI, 61 *Verso*, L. 3

26. *ra*

    i. Plate XXIX, 43 *Recto*, L.4

    ii. Plate XXXIV, 50 *Recto*, L.3

27. *la*

    i. Plate XLI, 61 *Verso*, L. 6.

    ii. Plate XLIV, 66 *Verso*, L. 3.

28. *va*

    i. Plate II, 1 *Verso*, L. 7.

    ii. Plate XXXIII, 49 *Verso*, L. 3.

29. *śa*

    i. Plate XXXIV, 51 *Recto*, L. 4.

    ii. Plate XXXIX, 58 *Recto*, LL. 2,3.

30. *ṣa*

    i. Plate XXIV, 36 *Recto*, L. 4

    ii. Plate XXXIX, 57 *Verso*, L. 2.

31. *sa*

    i. Plate XXXIX, 58 *Recto*, L. 2.

    ii. Plate XL, 59 *Recto*, L. 2.

32. *ha*

    i. Plate VII, 8 *Verso*, L. 3.

    ii. Plate XXIII, 34 *Recto*, L. 5.

Jihvāmūlīya

i. Plate VI, 8 *Recto*, LL. 3, 4.

ii. Plate VIII, 10 *Verso*, LL. 2.

Upadhmānīya

i. Plate V, 5 *Verso*, L. 1.

ii. Plate XXV, 37 *Recto*, L. 4.

iii. Plate XXV, 36 *Verso*, L.5.

Visarga

Plate V, 6 *Verso*, L. 4.

Virāma

i. Plate XXV, 37 *Verso*, L. 6.

ii. Plate XXXVIII, 56 *Verso*, LL. 2,3,4.

iii. Plate XLV, 67 *Recto*, L. 6.

TABLE 'C'

MEDIAL VOWELS

1. *ā*
   - i. Plate XXXII, 47 *Verso*, L.3
   - ii. Plate XXV, 37, *Recto*, L. 5.
   - iii. Plate XLIV, 66 *Recto*, LL. 2,3.
   - iv. Plate V, 5 *Verso*, L. 2.
   - v. Plate XII, 16 *Recto*, L. 6.

2. *i*
   - i. Plate XXV, 37 *Verso*, L. 6
   - ii. Plate XXV, 37 *Recto*, L. 7.

3. *ī*
   - i. Plate XVII, 25 *Verso*, L. 10.
   - ii. Plate XXV, 37 *Recto*, L. 6.

4. *u*
   - i. Plate IV, 4 *Verso*, L. 6.
   - ii. Plate XXXL, 51 *Verso*, L. 4.
   - iii. Plate XXXL, 51 *Verso*, LL. 3,8.

5. *ū*
   - i. Plate XXXIV, 50 *Recto*, L. 4.
   - ii. Plate XLII, 63 *Recto*, L. 2.
   - iii. Plate XXV, 37 *Recto*, L. 7.

6. *hṛ*
   - i. Plate I, 30 *Recto*, L. 3.
   - ii. Plate XLII, 63 *Verso*, L. 6.

7. *rū*
   - i. Plate VI, 8 *Recto*, L. 2.
   - ii. Plate XLIV, 66 *Verso*, LL. 2, 3.

8. *brū*
   - i. Plate XXV, 37 *Recto*, L. 7

9. *e*
   - i. Plate XLII, 65 *Recto*, L. 6.
   - ii. Plate IV, 5 *Recto*, LL. 7,3.
   - iii. Plate XXXIV, 50 *Recto*, L. 3.

10. *ai*
    - i. Plate XII, 16 *Verso*, LL. 3, 4.
    - ii. Plate IV, 4 *Recto*, L. 2.
    - iii. Plate XLIII, 65 *Recto*, L. 3.

11. *o*
    i. Plate XXXVIII, 56 *Recto*, L. 2.
    ii. Plate XIV, 20 *Verso*, L. 3.
    iii. Plate XXXL, 51 *Verso*, LL. 2, 3.

12. *au*
    i. Plate XXIII, 34 *Recto*, L. 2.
    ii. Plate XXIII, 33 *Verso*, L. 6.
    iii. Plate XLII, 63 *Verso*, L. 6.

TABLE 'D'

LIGATURES

1. *kṛ*

   Plate XLII, 63 *Recto*, L. 3.

2. *ku*

   Plate XXXII, 47 *Recto*, L. 9.

3. *kra.*

   Plate XLII, 63 *Recto*, L. 5.

4. *śu*

   Plate XLII, 63 *Verso*, L. 5.

5. *jña*
   i. Plate XIII, 18 *Verso*, L. 7.
   ii. Plate IV, 5 *Recto*, L. 7.

6. *ṣṭa*
   i. Plate XXXIV, 50 *Recto*, L. 5.
   ii. Plate XXIII, 34 *Recto*, L. 2.
   iii. Plate VIII, 11 *Recto*, L. 6.
   iv. Plate VII, 9 *Recto*, L. 7.

7. *ṣṭha*

   Plate XXXIV, 50 *Recto*, L. 4.

8. *stha*
   i. Plate IV, 4 *Verso*, L. 8.
   ii. Plate VIII, 10 *Verso*, L. 6.
   iii. Plate XII, 16 *Verso*, L. 5.

9. *rtha*

   Plate VIII, 10 *Recto*, L. 4.

10. *rdi*

    Plate IV, 4 *Recto*, L. 5.

11. *rva*

  i. Plate IV, 4 *Recto*, L. 4.

  ii. Plate IV, 4 *Verso*, L. 5.

12. *rya*

  Plate III, 3 *Recto*, L. 4.

13. *rdha*

  Plate VI, 7 *Verso*, L. 2.

14. *rṇa*

  i. Plate XIV, 20 *Recto*, L. 6.

  ii. Plate XIII, 18 *Recto*, L. 4.

15. *pra*

  Plate XXXII, 47 *Recto*, L. 6.

16. *krī*

  Plate XXXIII, 49 *Recto*, L. 4.

17. *trai*

  Plate VII, 9 *Verso*, L. 4.

18. *sya*

  Plate V, 5 *Verso*, L. 6.

19. *tya*

  Plate VI, 8 *Recto*, L. 3.

20. *pta*

  Plate XXXIX, 58 *Recto*, L. 2.

21. *sta*

  Plate XLII, 63 *Recto*, L. 6.

22. *kṣa*

  i. Plate V, 5 *Verso*, L.5.

  ii. Plate IV, 5 *Recto*, L.7

23. *tva*

  Plate IV, 5 *Recto*, L.7.

24. *jjha*

  Plate IV, 4 *Recto*, LL.5, 6.

25. *ñca*

  Plate V, 5 *Verso*, L.1.

CHAPTER IV

THE AGE OF THE BAKHSHALI MANUSCRIPT

IV

THE AGE OF THE BAKSHALI MANUSCRIPT

There is a lot of controversy with regard to the age of the *Bakhshāli Manuscript*. The Controversy is due to the fact that no where in the preserved part of the Manuscript do we have any indication of the date of its composition. Its begining is completely lost and the colophon is only partly preserved. In the colophon the name of the father of the author has only been preserved.[1] The work contained in the *Bakhshāli Manuscript* is not referred to or quoted by any known Indian or foreign mathematician. Thus, it is has been sought to determine the age of the Manuscript on the basis, of the script and the language in which the manuscript is written and the subject matter contained in the text.

The *Bakhshāli Manuscript* is written in the *Śāradā* script, the direct decendent of the north western branch of the *Brāhmi*, which makes its appearance in the 9th century A.D. Its continued use in *Gandhāra* to which our Manuscript belongs even as late as the fifteenth century is attestted to by the Peshawar Museum Inscription of Vaṇhadaka, which is dated Saturday, the 13th Lunar day of the bright half of Kārtik in the *Laukika Saṁvat* 538, Corresponding to 17th October 1461. The other important *Sarada* inscriptions of the region belong to the Brāhmaṇ-Shahi rulers Bhimadeva and Jayapāladeva who ruled over Gandhāra in the tenth and eleventh centuries. The period of Brāhmaṇ-Shahi rule, in Gandhāra witnessed a period in which Hindu civilization and Brāhmaṇical learning flourished. An insight into the flourishing Shahi rule the state of learning and enormous resources of the shahi kings is furnished, besides the inscriptions referred to above by Albiruni who discusses in detail the flourishing

rule of the Brāhmaṇ-Shahi rulers and their heroic struggle against the Ghaznav-ide-invaders. While discussing the age of the Manuscript, Hoernle[2] remarks, "The composition of Hindu work on arithmetic such as that contained in the Bakhshāli Manuscript seems necessarily to presuppose a country and a period in which Hindu civilization and Brahmanical learning flourished". Since the period of the Brāhmaṇ-Shahi rulers was conducive to the flourishing of Brāhmaṇical learning in Gandhāra, it would not be wrong to assume that the Bakhshāli Manuscript was the product of this age. The consideration of the script in which the Manuscript is written points to the same. The knowledge of the Śāradā script is now far well advanced than it was at the time when Hoernle and Buhler wrote. The two authorities on Indian paleography based their conclusions on the basis of very few Śāradā epigraphic records that were available and accessible to them. Since the time of two scholars a pretty large number of Śāradā epigraphic records have come to light. The script employed in the Śāradā records both literary and epigraphic, available to date has now been scientifically analysed and the knowledge of the alphabet put on a very sound basis by Dr. B.K. Koul Deambi.

The script of our text has been discussed in detail by Dr. Kaye[3] and Dr. B.K. Koul Demabi[4]. The paleographic peculiarities displayed by it have been discussed in detail above. It has been shown that the characters employed in our Manuscript bear close resemblance to those employed in the Śāradā epigraphic records of Gandhāra and the neighbouring regions of Kashmir and Himachal Pradesh belonging to 11th and 12th centuries. Thus the consideration of the script points to eleventh and twelfth centuries as the probable age of our Manuscript.

The conclusions arrived at on the basis of the script, employed in the Manuscript, are corroborated by the consideration of the language in which the

text is written. The Manuscript is written in Sanskrit language full of varnacular infuences. The features of the language displayed by our text which have been discussed in detail above were peculiar to not a few Sanskrit records of North-Western India of this age. This is amply borne out by the Buddhist text discovered from Gilgit, called the *Gilgit Manuscripts* edited by Nalanaksh Datta, and the Śāradā epigraphic records discovered from Kashmir and Chamba.

G.R. Kaye[5] considers the *Bakhshāli Manuscript* and the work contained in it as contemporaneous. However, Hoernle[6] and B.B. Datta[7] regard the *Bakhshāli Manuscript* and the work contained in it as belonging to two different ages. While conceding a later date for the Manuscript, they regard the Bakhshāli work to belong to much earlier period. Thus Hoernle[8] observes, "Quite distinct from the question of the manuscript, is that of the age of the work contained in it. There is every reason to believe that the Bakhshāli arithmetic is of a very considerably ealier date than the Manuscript in which it has come down to us. I am disposed to believe that the composition of the former must be referred to the earliest centuries of our era, and that it may date from the 3rd or 4th century A.D." This estimation about the age of the original Bakhshāli work has been accepted as agreeable by Buhler[9], Cantor[10], Cajori[11] and B.B. Datta[12]. B.B. Datta asserts that there is internal evidence of unquestionable value to show that the Bakhshāli mathematics can not belong to so late a period as assigned to it by Kaye. Datta further regards the Bakhshāli work not as *Karana* work as suggested by Hoerne, but a commentary on an earlier work of *Karana* type, "the manner of its composition and particularly the very elaborate, rather over elaborated details with which the various workings of the solutions and most carefully recorded without trying to avoid even unnecessary repititions strongly tend to such a conclusion."[13]

The following arguments tend to support the conclusion that the Bakhshāli Manuscript contians an original mathematical work of an earlier date —

1. The method of exposition found in our text differs considerably from what is now commonly met with in other Indian mathematical treatises. Brahmgupta (628 A.D.)[14] gives very few *udāharaṇa* (examples) in illustrations of a limited number of his rules, but not their solution. This want has been amply made up by his eminent commentator Pṛthudakasvāmi (860 A.D.) who has supplied sufficient number of illustrative examples with solution under each rule. Mahāvīra (c. 850 A.D.)[15] gives a copious number of examples for each rule. He calls them *uddeśaka*. But he does not give the solution, too. The first writer to give *nyāsa* (statements) as well as answers of his *udāharaṇa* (examples) is Srīdhara (c. 850 A.D.)[16]. Then comes Bāskara (1114 A.D.)[17]. These writers have not recorded workings of their solution. The proofs or verification given in our text of the solution of examples is unique to our text. No other Indian writer is known to have given any verification of the solution of their examples. This scheme of exposition is not found even in works of the later commentators.

Thus the scheme of exposition employed in the *Bakhshāli Manuscript* is also considered indicative of the antiquity of the work it contains.

2. The rule for determining approximate value of a non-square number discussed in detail below in Chapter V, is attributed to Greek Heron (c. 200 A.D.)[18] and has been restated by the Arab-al-Has-Sara[19] Kaye[20] observes that this rule was never used in the early Indian works to any extant whereas the Bakhshāli text employs it for a comparatively large number of examples and applies this rule to second approximations in a very thourough manner. Datta[21] however asserts that it was known to the second order of approximation to the ancient Hindus at a much earlier period. Datta's references in this regard to earlier works

have been discussed elsewhere in chapter VII in this thesis. It may be pointed out that while Datta may be right in his assertion that the method of finding out the square root of a non-square number was known to Indians at a much early period, but we agree with Kaye that the particular method, employed in the Bakhshāli work to find out the square-root of a non-square number is not found in other Indian texts. Śrīdhara[22] and Bhāskara[23] give the following method of finding the square-root of a non-square number, which differs considerably from the method employed in our text. 'Multiply the quantity whose square-root cannot be found by any large square number, take the square-root of the product — leaving out of account the remainder — and divide by the square-root of the multiplier.

$$\text{e.g. } \sqrt{41} = \sqrt{41 \times 1000000} \div 1000 \approx 6.403$$

3. The negative sign represented by a cross (+) used in the Manuscript is regarded as a mark of antiquity. The later writers of the Indian mathematics use a dot (.) to indicate the negative quantity. It is surmised by Hoernle[24] followed by Datta[25] that the negative symbol + stands for *ka* as the sign for ka in the Ashokan Brāhmi resembles a cross (+). They indentify the symbol with *ka* and take it to as an abbreviation of *kṣaya*, the word used for the operation of subtraction in our text. Even if the conjucture of Hoernle and Datta be regarded as sound, it remains to be explained why the old Asokan sign was used for a particular symbol in a Śāradā Manuscript, when besides the characters all the numerical notations used in the text belong to the Śāradā alphabet. Hoernle's assertion that Brāhmi *ka* undergoes little change and retains its archaic form in Śāradā is not based on fact. The Śāradā *ka* differs materially from the old Brāhmi *ka* and even in the ligatures where *ka* is a first member and retains its early form.

The latter difers considerably in shape from the cross (+) used to indicate the negative sign in our text. The origin of this sign remains obscure even if it is conceded that the use of this particular symbol is a mark of antiquity.

4. The method of finding the least common multiple followed in our text and discussed ahead in chapter V of this thesis is not found in the works of Āryabhaṭa, Brahmagupta and Bhāskara. The same method is followed in the *Gaṇita-Sāra-Saṁgraha* of Mahāvīra (c. 850 A.D.)[26].

5. The arithmetical notation used in our Manuscript is the decimal place-value notation. The exclusive use of this notation is noteworthy as word numerals in place of the decimal place value notation are usually used in all available mathematical treatises. The only exception is the *Āryabhaṭiya* of Āryabhaṭa (499 A.D.) where decimal place value notation has been used as in our text.

The technical terms which are generally employed in the Bakhshāli mathematics are mostly the same as in other Hindu treatises on mathematics. There are a few terms which distinguish it at once from the rest. For example, the common Indian term for the reduction of fractions to a common denominator is *savarṇana,* which means 'making of the same class'. But according to Bakhshāli mathematics it should be *sadṛśi-karaṇa,* which means 'making similar' or *hara-sāmya-karaṇa,* which means 'making the denominators equal.' There tersms *'Sadṛśam-kriyate'*[27], *hara-sāmya kriyate*[28] and *sadṛśa kr(te)*[29] occur in different examples in Bakhshāli mathematics. The term *savarṇa* is found only once in the Bakhshāli mathematics as forming a part of another compound word, *kalāsavarṇa,* which refers to the fraction in general or at least to a particular kind of it. The same term reappears in the sense of general fraction in the *Gaṇita-sāra-saṁgraha* of Mahāvīra[30] and a nearly equal term in the Triśatika of Śrīdhara[31]. Āryabhaṭa (499 A.D.) has also adopted this term savarṇana and all

later Indian Mathematicians have also used this term. So its absence from *Bakhshāli Manuscript* clearly shows that the time of the manuscript is a period anterior to the fifth century of Christian era. Again, the usual Indian term for the series, from the fifth century A.D., is *średhi*, meaning 'series' but Bakhshāli mathematics has a different term *'varga'* which means 'group' for the term 'series'. Similarly, there are more terms used in the Bakhshāli mathematics, those strongly refer the time of the work before fifth century A.D. For example,

I. The technical term for the statement of a problem in Bakhshāli mathematics is frequently 'sthāpana' and occasionally *nyāsa* or *nyāsa-sthāpana*, while in the later Indian works it is only *nyāsa*. Now the term *nyāsa-sthāpana* is surplus to requirements, for both constituents of it bear the same significance, so that either would have been sufficient. Its occurance, as also that of *sthāpana* in the place of *nyāsa*, very likely implies that the Bakhshāli work must be referred to a period of transition before the introduction of the modern term nyāsa.

II. The sub-section in the Bakhshāli work dealing with the mixture of golds of different varieties is called Suvarṇa-kṣaya[32] which means 'loss of gold'; in the *Līlāvatī*[33] it is called *suvarṇa-gaṇita* which means 'computations relating to gold'; in the *Gaṇita-sāra-Saṁgraha*[34], *suvarṇa kuṭṭikāra* or *suvarṇa-gaṇita*, and in the *'Trisatika'*[35], *suvarṇa-varṇa-parijñāna*.

III. The rules dealing with 'interest' in our text, is called *hundikāsamānayana* sūtras[36], while the corresponding terms in all other works are different.

6. The Bakhshāli mathematics is particularly characterised by the absence of any kind of algebraic symbols and notations. Though it shows a fair degree of progress in the science of algebra, there is not even a specific notation to represent the unknown quantity. This lack of symbolism has given rise to a certain amount of misunderstanding and at times has led to the adoption of the

method of 'false position' or 'supposition' for the solution of the equation. Similarly the work has no special signs for the arithmetical operations, too. If any operation is intended, it is generally indicated by placing the abbreviation (initial syllable) of a Sanskrit word of that import after, occasionally before, the quantity affected. Thus the operation of addition is indicated by '*yu*' (an abbreviation for yuta, meaning 'added' subtraction by + which is probably from '*ksa*' (abbreviated from *kṣaya* 'diminished'), multiplication by '*gu*' (abbreviation of *guṇa* or *guṇita*, meaning multiplied') and the division by 'bhā' (abbreviated from *bhāga* or *bhājita*, meaning 'divided'). This principle of choosing abbreviations of the words of respective imports as the signs of the first four fundamental arithmetical operations, as found in the Bakhshāli work is not met with in other Indian treatises on mathematics, or indeed in any early mathematics. Similarly, the square root of a quantity is indicated by writing after it '*mū*', which is an abbreviation for mūla, meaning "root" while in the rest of the Indian mathematics, it is indicated by 'ka', an abbreviation from *karani*, meaning "surd". The application of approximate square-root formula is not found expressly stated in its entirety in any Indian treatise on mathematics except in Bakhshāli mathematics. From the time of Āryabhaṭa (499 A.D.) the approximate square-root rule is not found anywhere onwards. But it appears to have been understood in India about the beginning of the christian era and in the few centuries preceding it.

Thus on the afore-discussed considerations the original Bakhshāli work appears to have been composed in the early centuries of the christian era.

REFERENCES

1. *Bakh. Ms.* folio, 50 recto, p. 237.
2. Hoernle, *I A.*, Vol. XVII, p. 36.
3. Kaye, *Bakh. Ms.* pp. 87-105.
4. Deambi, B.K. Koul, *op. cit.*, pp. 4, 67-91.
5. Kaye, *Bakh. Ms.* pp 74-84.
6. Hoernle, *I A.*, Vol. XVII, p. 36
7. Datta, B.B., 'The Bakhshāli Mathematics', *Bulletin of the Calcutta Mathematical Society*, Vol. XXI, 1929, pp. 1-60.
8. Hoernle, *I A.* Vol. XVII, p. 36.
9. Buhler, *Indian Paleography*, p. 82
10. Cantor, M. *Geschichte der Mathematik*, 1, part 3, p. 598.
11. Cajori, F., *History of Mathematics*, Second edition, Boston, 1922, p. 85.
12. Datta, BCMS, Vol. XXI;, pp. 2-3
13. *op. cit.*, p. 4.
14. *Brāhma-sphuṭa-siddhānta, op. cit.*
15. *Gaṇita-Sāra-Saṁgraha, op. cit.*
16. *Pāṭi-gaṇita, op. cit.*
17. *Līlāvatī, op. cit.*
18. Heath, T.L., *A History of Greek Mathematics*, Vol. II, Oxford, 1921, p. 324.
19. Smith, D.E., *A History of Mathematics*, Vol. II, Dover-publications, 1958, p. 254.
20. Kaye, *Bakh. M S.* pp. 30-31.
21. Datta, B.B., 'Hindu Contribution to Mathematics', *Bulletin of the Mathematical Association*, University of Allahabad, Vol. I, 1927-28, p. 6.

22. *Pāti-gaṇita, op. cit*, 46.

23. *Līlāvatī, op. cit*, 138.

24. Hoernle, *I A*. Vol. XVII, p. 34.

25. Datta, *BCMS*, Vol. XXI, pp. 17-18.

26. *Gaṇita-sāra-Saṁgraha*, iii, 56.

    cf. Datta, *BMA*, vol. I.

27. *Bakh,. MS*. folio, Iverso, p. 168.

28. *Bakh. MS*. folio, 17 recto, p. 210.

29. *Bakh MS*. folio, 30 verso, p. 213.

30. *Gaṇita-Sāra-Saṁgraha*, iii, 1.

    cf. Datta, *BCMS*, vol XXI, p. 37.

31. *Triśatikā*, pp 7, 12. In this book the fraction is more commonly called *bhinna*, which literally means 'broken-part'; it is also called *kalāsa-varṇana*, cf. Datta, *BCMS*, vol. XXI, p. 37.

32. *Bakh. MS* folio, 16 verso, p. 209.

33. *Līlāvatī*, p. 24.

34. *Gaṇita-Sāra-Saṁgraha*, VI, 169.

35. *Triśatikā*, p. 25.

36. *Bakh. MS*. folio, 67 recto, p. 188.

CHAPTER V

ANALYSIS OF CONTENTS

V

ANALYSIS OF CONTENTS

The *Bakhshāli Manuscript* represents a work on mathematics containing rules pertaining to mathematical problems called *Sūtras* in the text, the examples illustrating the sūtras and the soluions accompanying the examples. The subject matter deals with Arithmetic and Algebra and occasionally with Geometry. The major part of the work deals with Arithmetic. However, as rightly pointed out by G.R. Kaye[1] although the work as arithmetical in form it would not be correct to describe it as a simple arithmetical in form or generalised arithmetic or algebra. The topics of discussion are found to include: Rule of three, fraction, square-root, arithmetical and geometrical progressions, income and expenditure, profit and loss, computation of gold, summation of series, simple equation, simultaneous linear equations, quadratic equation, indeterminate equation of second degree and miscellaneous problems.

In this chapter, we attempt an analysis of the contents of the Manuscript as far as it is possible to make out of the preserved portion of the text. Before we attempt an analysis of the contents of the work which are predominently mathematical in nature, it would be worthwhile to study the exposition and method employed in the text.

Scheme of Exposition :-

The subject - matter is divided in *Sūtras* (Rules). These *Sūtras* are expressed in very concise language, but are fully explained by means of *Udāharaṇas* (examples). There are generally two *udāharaṇs* after each *Sūtra*, but sometimes there are many. The rules and examples are written in verse; the explanations, solutions and everything else are in prose. The metre used is the *Śloka*. The *Sūtra* and *Udāharaṇa* are followed by *Sthāpanā* (statement). At the stage of *Sthāpanā* the *Udāharaṇa* is repeated in the form of a notation in figures. This is followed by *Karaṇa* (the solution). At last comes the *Pratyayam* (the verification).

The end of each *Sūtra* is marked after the last example by the device (fig. i) and the number of the *Sūtra* is also given at the end. The grouping of the sets of the figures is done as under:-

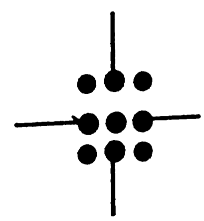


fig. i

The numbers are generally put in cells. An integer is either written in a cell e.g. $\boxed{2}$ or in vertical bars, e.g., $|4|$. Fractions are also put in cells e.g., $\boxed{\begin{matrix}1\\4\end{matrix}}$ which means $\frac{1}{4}$ and groups of fractions are also put in cells e.g.:-

I. $$\begin{array}{|cc|cc|} 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 13 \\ \hline \end{array}$$

which means $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{13}$;

II.	1 1 3 1 4 1 5	Which means $(1+\frac{1}{3})\ (1+\frac{1}{4})\ (1+\frac{1}{5})$;

and III.	40 1 1 3 + 1 4 + 1 5 +	which means 40 $(1-\frac{1}{3})\ (1-\frac{1}{4})\ (1-\frac{1}{5})$; ('+' stands for the negative sign, see below).

FUNDAMENTAL OPERATIONS.

Any particular operation intended is generally indicated by placing the abbreviation (initial syllable) of a Sanskrit word of that import after, occasionally before, the quantity affected. Thus addition is indicated by writing *yu* (abbreviated for *yuta* "added") before or after the additive quantity and placing the latter either by the side of or below the other quantity;

e.g., 11 5 *yu* or 11 *yu* 5, means 11 + 5.

Besides the above, the operation of addition is indicated by servral other ways, some of which are given below:

i. 1|1 *yutaṁ* |2|[2]

which means 1 and 1 added is 2

or 1 + 1 = 2.

ii. 12 *dvi - yutaṁ* |14|[3]

which means 2 added to 12 is 14

or 12 + 2 = 14

iii. |20| 40 |60| 80| *evaṁ* 200|[4]

which means 20, 40, 60, 80 thus 200

or 20 + 40 + 60 + 80 = 200

iv. | 1 | 4 | 9 | 16 | *eṣa yuti* $\boxed{30}$ [5]

which means 1, 4, 9, 16 the sum of these is 30

or 1 + 4 + 9 + 16 = 30.

v. | 90 | 80 | 75 | 72 *chaturnām yoga* $\boxed{317}$ [6]

which means 90, 80, 75, 72 sum of the four is 317.

or 90 + 80 + 75 + 72 = 317.

vi. $\boxed{\begin{matrix}10\\3\end{matrix}}$ *sa rūpam* $\boxed{\begin{matrix}13\\3\end{matrix}}$ [7]

which means $\frac{10}{3}$ plus *rūpa* (unity) is $\frac{13}{3}$

or $\frac{10}{3} + 1 = \frac{13}{3}$

vii. $\boxed{\begin{matrix}3\\5\end{matrix}}$ *rūpam dadya* $\boxed{\begin{matrix}8\\5\end{matrix}}$ [8]

which means unity given to $\frac{3}{5}$ is $\frac{8}{5}$.

or $\frac{3}{5} + 1 = \frac{8}{5}$.

viii. | 120 | 90 | 80 | 75 | 72 | *eṣam yoga krite jātā* 437 [9]

which means 120, 90, 80, 75, 72 sum of these is made 437

or 120 + 90 + 80 = 75 = 72 = 437.

ix. $\boxed{\begin{matrix}120\\30\\80\\75\\72\end{matrix}}$ *evaṁ* 377[10]

which means 120, 30, 80, 75, 72 thus 377.

or 120 + 30 + 80 + 75 + 72 = 377

x. $\boxed{\begin{matrix}5 & \acute{s}e & 1\\1 & & 16\end{matrix}}$ $\boxed{10 \quad \acute{s}e \quad \frac{15}{16}}$ *evaṁ* 16[11]

which means $5\frac{1}{16}$, $10\frac{15}{16}$, thus 16

or $5\frac{1}{16} + 10\frac{15}{16} = 16$

xi. $\lfloor 10 | 30 | 90 \rfloor$ *Ekatram* $\left|\begin{array}{c}130\\1\end{array}\right|$ [12]

which means 10, 30, 90 together $\frac{130}{1}$

or $10 + 30 + 90 = 130$.

xii. $\frac{45}{2}$ *sārdha traya yutam* $\frac{52}{2}$ [13]

which means $\frac{45}{2}$ with $3\frac{1}{2}$ added is $\frac{52}{2}$

or $\frac{45}{2} + \frac{7}{2} = \frac{52}{2}$

The operation of subtraction is indicated by writing the negative sign indicated by the symbol '+' after the subtractive quantity and placing the latter beside or below the other quantity :

e.g. 11 7+ means 11-7,

and $\left|\begin{array}{c}1\\1\\3+\end{array}\right|$ means $1 - \frac{1}{3}$.

There are several other ways to indicate the operation of subtraction. Some of them are given below:-

i. | 5 | 9 | *viśeṣam* $\lfloor 4 \rfloor$ [14]

means 5, 9 the difference is 4.

or 9-5 = 4.

ii. $\lfloor 5 \rfloor$ $\lfloor 3 \rfloor$ *rahitam jātaṁ* $\lfloor 2 \rfloor$ [15]

means 5, 3 subtracted, 2 is produced.

or 5 - 3 = 2

iii. | 6 | 3 | *śuddhi* | 3 | [16]

which means 6, 3 the difference is 3

or 6 - 3 = 3

iv. $\lfloor 3 \rfloor$ $\lfloor 7 \rfloor$ *viśoddhya* $\lfloor 4 \rfloor$ [17]

which means 3, 7 having subtacted give 4 .

or 7 - 3 = 4

v. $\boxed{42}$ *trayūṇāṁ* $\boxed{39}$ [18]

which means 42 less by three is 39.

or 42 - 3 = 39.

vi. $\begin{array}{|c|c|}77 & 294 \\ 11 & 11\end{array}$ *Pātya śeṣaṁ* $\begin{array}{|c|}217 \\ 11\end{array}$ [19]

which means $\frac{77}{11}$, $\frac{294}{11}$ having subtracted, the difference is $\frac{217}{11}$

$$\text{or } \frac{294}{11} - \frac{77}{11} = \frac{217}{11}$$

Multiplication is indicated by placing one quantity beside the other e.g.

i. $\frac{5}{8}$ 32 (= $\frac{5}{8}$ x 32 = 29),

ii. By *gu* (abbreviation of *guṇa* or *guṇita*, meaning "multiplied" e.g.

$\begin{array}{|cc|}2 & 40 \\ 5 & 1\end{array}$ *guṇita jātaṁ* 16

or $\frac{2}{5} \times \frac{40}{1} = 16$

Some other examples of multiplications are :-

iii. $\boxed{2}$ *dviguṇaṁ* $\boxed{4}$ [20]

which means 2 multiplied by 2 is 4

iv. $\boxed{10}$ *dviguṇaṁ* $\boxed{20}$ [21]

which means 10 multiplied by 2 is 20.

v. $\boxed{30}$ *aṣṭa guṇaṁ* $\boxed{240}$ [22]

which means 30 multiplied by eight is 240.

vi. $\begin{array}{|c|c|}7 & 18 \\ 9 & 1\end{array}$ *guṇitam jātaṁ* $\boxed{14}$ [23]

which means $\frac{7}{9}$ multiplied by $\frac{18}{1}$, 14 is produced.

vii. $\begin{array}{|c|c|c|c|}8 & 2 & 5 & 4 \\ 1 & 3 & 6 & 5\end{array}$ *guṇitam jātaṁ* $\begin{array}{|c|}32 \\ 9\end{array}$ [24]

which means 8, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{4}{5}$ multiplied, $\frac{32}{9}$ is produced.

$$\text{or } 8 \times \frac{2}{3} \times \frac{5}{6} \times \frac{4}{5} = \frac{32}{9}$$

vi. $\left|\begin{array}{c|c} 6 & 1 \\ 1 & 1 \\ & 4+ \end{array}\right|$ *anena guṇitaṁ jātaṁ* $\left|\begin{array}{c} 4 \\ 1 \\ 2 \end{array}\right|$ [25]

which means 6 multiplied by $1 - \frac{1}{4}$, $4\frac{1}{2}$ is produced

$$\text{or} \quad 6 \times \frac{3}{4} = 4\frac{1}{2} \quad \text{or} \quad \frac{9}{2} = 4\frac{1}{2}$$

viii. $\left|\begin{array}{c|c} 3 & 4 \end{array}\right|$ *abhyāsaṁ* $\left|\begin{array}{c} 12 \end{array}\right|$ [26]

which means 3, 4 the product is 12.

$$\text{or } 3 \times 4 = 12.$$

viii. $\left|\begin{array}{c} 8 \end{array}\right|$ *ātma guṇaṁ* $\left|\begin{array}{c} 64 \end{array}\right|$ [27]

which means 8 multiplied by itself is 64.

$$\text{or } 8 \times 8 = 64.$$

ix. $\left|\begin{array}{c|c|c|c} 10 & 5 & 5 & 5 \\ 1 & 4 & 4 & 4 \end{array}\right|$ *guṇitaṁ jātaṁ* $\left|\begin{array}{c} 1250 \\ 64 \end{array}\right|$ [28]

which means 10, $\frac{5}{4}$, $\frac{5}{4}$, $\frac{5}{4}$ multiplied is $\frac{1250}{64}$

$$\text{or} \quad 10 \times \frac{5}{4} \times \frac{5}{4} \times \frac{5}{4} = \frac{1250}{64}.$$

x. $\left|\begin{array}{c|c} 880 & 964 \\ 84 & 168 \end{array}\right|$ *guṇitajātaṁ* $\left|\begin{array}{c} 848320 \\ 14112 \end{array}\right|$ [29]

which means $\frac{880}{84}$, $\frac{964}{168}$ multiplied is $\frac{848320}{14112}$

$$\text{or} \quad \frac{990}{84} \times \frac{964}{168} = \frac{848320}{14112}.$$

xi. $\left|\begin{array}{c|c} 405280 & 444004 \\ 38724 & 77448 \end{array}\right|$ *saṁguṇyajātaṁ* $\left|\begin{array}{c} 179945941120 \\ 2999096352 \end{array}\right|$ [30]

which means $\frac{405280}{38724}$, $\frac{444004}{77448}$ having

multiplied these together, $\frac{179945941120}{2999096352}$ is produced.

$$\text{or} \quad \frac{405280}{38724} \times \frac{444004}{77448} = \frac{179945941120}{2999096352}.$$

The use of the word *parasparakṛtaṁ* (making together) for multiplication is also used in the Manuscript.

e.g., | 4 | 6 | 7 | *tata śeṣaṁ paraspara kritaṁ guṇita jātaṁ* $\lfloor 168 \rfloor$ [31]

4, 6, 7 after making together is 168.

The operation of division is indicated i. by placing one quantity under another without a line between them; e.g., $\begin{matrix}5\\8\end{matrix}$ $(= \frac{5}{8})$.

ii. By *bhā* (abbreviated form of *bhāga* or *bhājita*, "divided"): e.g.-

1	1	1	1		36
1	1	1	1	*bhā*	
2+	3	4+	5		1

which means $\dfrac{36}{(1-\frac{1}{2})\ (1+\frac{1}{3})\ (1-\frac{1}{4})\ (1+\frac{1}{5})}$.

There are some other forms of abbreviations and word-forms, used in the Manuscript to indicate 'Division', those we shall discuss in the following examples-

i. | 168 / 4 | 168 / 6 | 168 / 7 | *labhadhān* 42 | 28 | 24 [32]

which means - the quotient of $\frac{168}{4}$, $\frac{168}{6}$, $\frac{168}{7}$ is 42, 28, 24

ii. $\lfloor 30 \rfloor$ *vibhaktaṁ* $\left|\begin{matrix}1\\30\end{matrix}\right|$ [33]

means 30 divided is $\frac{1}{30}$.

iii. $\lfloor 10 \rfloor$ $\lfloor 3 \rfloor$ *vibhaktaṁ* $\left|\begin{matrix}10\\3\end{matrix}\right|$ [34]

means 10, 3 : having divided $\frac{10}{3}$ (is produced)

or 10, 3 divided = $\frac{10}{3}$

iv. $\left|\begin{matrix}10225\\32800\end{matrix}\right|$ *dalita* $\left|\begin{matrix}10225\\65600\end{matrix}\right|$ [35]

means $\frac{10225}{32800}$ halved is $\frac{10225}{65600}$ or $\frac{10225}{32800} \times \frac{1}{2} = \frac{10225}{65600}$

v. $\begin{array}{|c|}75\\15\end{array}$ *vartyaṁ* $\begin{array}{|c|}5\\1\end{array}$ [36]

which means $\frac{75}{15}$ reduced gives $\frac{5}{1}$

or $\frac{75}{15} = 5.$

vi. $\begin{array}{|c|}90\\15\end{array}$ *vartyaṁ jātaṁ* $\begin{array}{|c|}6\\1\end{array}$ [37]

which means $\frac{90}{15}$ reduced, $\frac{6}{1}$ is produced

or $\frac{90}{15} = 6$

vii. $\begin{array}{|c|}473500\\947\end{array}$ vartita jātā phalaṁ 500 [38]

which means $\frac{473500}{947}$ reduced, 500 is produced

or $\frac{473500}{947} = 500$

viii. $\begin{array}{|c|}798\\1463\end{array}$ projjhya 798 [39]

which means having discarded its denominator $\frac{798}{1463}$ becomes 798.

ix. $\begin{array}{|c|}60\end{array}$ *anena driṣyam bhājitaṁ* $\begin{array}{|cc|}1 & 300\\60 & 1\end{array}$ *jātā* $\begin{array}{|c|}5\end{array}$ [40]

which means by this 60 the known quantity 300 is divided and 5 is produced.

or $\frac{1}{60} \times 300 = 5.$

Hence addition is indicated by yu (for yuta), subtraction by + (standing perhaps for kṣa for kṣaya), multiplication by *gu* (for *guṇa* or *guṇita*) and division by *bhā* (for *bhāga*). The whole operation, thus put between lines or vertical bars and the result is set down outside, introduced by *pha* (for *phala*) or sometimes written full e.g. $\begin{array}{|c|}5\ \ 7\ \text{yu}\end{array}$ *pha* 12, which means 5 + 7 = 12.

The square is represented by *va* (for *Varga*). The word *varga* literally means 'rows', but in mathematics, it ordinarily denotes "the product of two equal numbers". e.g. $\begin{array}{|c|}384\end{array}$ *asya varga* $\begin{array}{|c|}147\ 456\end{array}$ [41] which means square of 384 is 147456

or 384 x 384 = 147456.

FRACTIONS

In the Bakshāli Manuscript, the knowledge of fractions can also be traced :

e.g. i. the fraction 3/8 is called *tryaṣṭa* [42] (three-eighths)

ii. the fraction $3\frac{3}{8}$ is called *trayastrayaṣṭa* [43] (three-three-eighths).

It is quite possible that there may have been many more examples of fractions, but it may not be preserved due to the mutiliated form of the Manuscript.

ARITHMETICAL NOTATION-WORD NUMERALS

In the *Bakhshāli Manuscript;* the arithmetical notation used is the decimal place-value notation. There is an evidence of the principal of word numeral system of arithmetical notation. there is the use of the words with numeral significance, such as *rūpa* [44] (=1), *rasa* [45] (=6), and *pāda* [46] (= $\frac{1}{4}$).

Again in an example [47], whose only object seems to be to express the following big number first in words, then in figures, thus:

'Śad-vimśas cha tri-pamchāśa ekona-trimśevacha|dvā-śā.... śad-virhśa chatuś'- chatvāl-imśasaptati| chatush-shashtina (va) ...... mśa narhtararh trir-āśiti ekavimśa .... pakam |' and in figures as 2 6 5 3 2 9 6 2 2 6 4 4 7 0 6 4 9 9 4 ..... 8 3 2 1 8.

SYMBOL FOR THE UNKNOWN QUANTITY

In the *Bakhshāli Manuscript*, the unknown quantity is called *'Yadricchā'* or *'Kāmika'*. These terms are mentioned in the examples like this-

i. *Yadricchā pinyase śūnye*[48].

ii. *Yadricchā vinyase śūnye.*[49].

that is 'putting any desired quantity in the vacant place.'

In another example it is stated that-

iii. *Kāmikam śūnye pinyastam*[50]

that is 'the desired quantity is put in the vacant place.'

Yet in another example, it is stated that _

iv. *Śunye sthāne rūpam datvā.*[51]

that is putting 1 in the vacant place'.

In these above mentioned examples, we find the use of common term *'śūnye'* i.e. 'vacant' or 'empty'. The unknown quantity in the text is referred to by the symbol, which is called *śūnye* (void or empty).

However B.B. Datta [52] does not regard the sign as a symbol for the unknown as has been supposed by Hoernle [53] and Kaye [54]. He regards it to be same as zero (śūnya) of the decimal arithmatical notation. He adds, 'Its use in connection with an algebraic equation in a sense

other than for arithmetical notation is simply to indicate that the quantity which should be there is absent or not known. Its place in the equation is left vacant and this is clearly indicated by putting the sign of emptiness there."[55] Datta's observation appears to be correct as the symbol is not used in the solutions and is often referred to as *śūnyasthāna* or empty place as indicated above.

The author of our text does not use any well defined notations for the unknown as is the case with other author on mathematics. The absense of well-defined notations often gives rise to some ambiguity in certain examples which contain more than one unknown and where the same symbol has been used to indicate all the unknowns. Thus e.g. in the following example[56] different unknowns have to be assumed at different vacant places.

(i)	0 1	5 1	*yu*	*mū*	0 1	*sa*	0 1	7+ 1	*mū*	0 1

which means $\sqrt{x+5} = S$ and $\sqrt{x-7} = t$.

Here the symbol ' $\frac{0}{1}$ ' stands for three different unknown quantities. It simply indicates in each case an unknown number.

In order to avoid such ambiguity, in one instance[57] which contains as many as five unknowns, abbreviations of ordinal numbers have been used to represent the unknows; e.g._

9 *pra°* 7 *dvi°*	7 *dvi°* 10 *tri°*	10 *tri°* 8 *cha°*	8 *cha°* 11 *paṁ°*	11 *paṁ°* 9 *pra°*

yutaṁ jātaṁ pratyaika 16| 17| 18| 19| 20

which means- $x_1 + x_2 = 16$, $x_2 + x_3 = 17$, $x_3 + x_4 = 18$, $x_4 + x_5 = 19$, $x_5 + x_1 = 20$.
(where $x_1 = 9$, $x_2 = 7$, $x_3 = 10$, $x_4 = 8$, $x_5 = 11$)

Here the abbreviations of the ordinal numbers such as *pra* (abbreviated from *Prathama*, 'first'), *dvi* (from *dvitīya*, 'second), *tri* (from tritīya, 'third'), *cha* (from *chaturtha*, 'fourth') and *paṁ* (from *paṁcama*, 'fifth') have been used to represent the unknowns.

The want of a proper symbol for the unknown eventually leads to the adoption of the method 'false position' or 'supposition' for solution of algebraic equations.

'ZERO' IN THE BAKHSHĀLI MANUSCRIPT

The Bakhshāli Manuscript employs a dot (·) for śūnya (zero) of the decimal arithmetical notation. As already stated, the same symbol 'zero' has also been used for the unknown quantity. For instance, in the followng example[58].

(i)	*ā°* 1 1	*u°* 1 1	*pa°* • 1	*labdhaṁ* 10 1

This is a 'statement' of an Arithmetical Progression, where *ādih* (the first term) is 1, *uttarah* (the common difference) is 1, *pada* (the number of terms) is unknown, and the quotient is 10. Here the symbol '●' simply indicates that the number of terms (*pada*) is unknown; i.e. that the place in the statement is empty. The symbol, here, does not enter into any operation. So, in this case and in the same cases, the symbol stands for an unknown quantity, since it enters into no arithmetical operation.

The zero symbol has been included among *'Saṁkhyā'* (the numbers) in the *Bakhshāli manuscript*. It is one of the ten fundamental figures of the decimal system of notation, (i.e. 0 1 2 3 4 5 6 7 8 9). It has been used in the calculation, too. For instance–

I. In addition –

(i) $\begin{array}{|c|c|c|c|c|} 20 & 90 & 80 & 75 & 72 \end{array}$ *eṣām yoga krite jāta* 437 [59]

which means 120 + 90 + 80 + 75 + 72 = 437

(ii) $\begin{array}{|c|} 10 \\ 3 \\ \hline \end{array}$ Sa *rūpaṁ* $\begin{array}{|c|} 13 \\ 3 \\ \hline \end{array}$ [60]

which means $\frac{10}{3} + 1 = \frac{13}{3}$

II. In subtraction –

(i) $\begin{array}{|c|} 40 \\ \hline \end{array}$ 16 *śeṣaṁ* $\begin{array}{|c|} 24 \\ \hline \end{array}$ [61]

which means 40 – 16 = 24.

(ii) $\begin{array}{|c|c|} 425042 & 400 \\ 19362 & 19362 \\ \hline \end{array}$ *śeṣaṁ* $\begin{array}{|c|} 424642 \\ 19362 \\ \hline \end{array}$ [62]

which means $\frac{425042}{19362} - \frac{400}{19362} = \frac{424642}{19362}$

III. In Multiplication –

(i) $\begin{array}{|c|} 10 \\ \hline \end{array}$ *dviguṇaṁ* $\begin{array}{|c|} 20 \\ \hline \end{array}$ [63]

which means 10 x 2 = 20

(ii) $\begin{array}{|c|} 30 \\ \hline \end{array}$ *aṣṭa guṇaṁ* $\begin{array}{|c|} 240 \\ \hline \end{array}$ [64]

which means 30 x 8 = 240.

IV In Division –

(i) $\left.\dfrac{473500}{947}\right|$ *vartita jātā phalaṁ* 500||[65]

which means $\dfrac{473500}{947} = 500$

(ii) $\left|\begin{array}{c|c} 1 & 300 \\ 60 & 1 \end{array}\right|$ *jātā* $\left|\,5\,\right|$[66]

which means $\dfrac{1}{60} \times \dfrac{300}{1} = 5.$

Similarly in the following example[67]__

" $\left|\begin{array}{c|c} 880 & 964 \\ 84 & 168 \end{array}\right|$ *guṇita jātaṁ* $\left|\begin{array}{c} 848320 \\ 14112 \end{array}\right|$

chatvāriṁśa prithak sthānām vargaṁ $\left|\,1600\,\right|$ *eṣa uparā pātya śeṣaṁ* $\left|\begin{array}{c} 846720 \\ 14112 \end{array}\right|$

vartya jātaṁ $\left|\,60\,\right|$"

which means – " $\dfrac{880}{84} \times \dfrac{964}{168} = \dfrac{848320}{14112}$,

The square of '40' different places is '1600'. On subtracting this from the number above (numerator), the remainder is $\dfrac{846720}{14112}$. On removal of the common factor, it becomes 60."

There are a large number of passages of this kind in the Manuscript, where 'zero' has been used in the calculation.

LOWEST COMMON MULTIPLE

In the *Bakhshāli Manuscript* the method to find out the lowest common multiple of fractions, is to reduce the fractions to the lowest common denominator before adding or subtracting. There are a few examples in the Manuscript those help us in understanding the above fact. These examples are as follows:-

I. In an example[68] it is required to find the sum of the fractions which though not preserved in the text can be restored as $\frac{2}{1}$, $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$, $1\frac{1}{5}$ After being reduced to the common denominator, the fractions become like this:-

$$\frac{120}{60}, \frac{90}{60}, \frac{80}{60}, \frac{75}{60}, \frac{72}{60}$$

Finally the sum is stated to be $\dfrac{437}{60}$

ii. In another example[69] to add the fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ these (fractions) are reduced to a common denominator, and take the following shape :-

$$\frac{120}{60} + \frac{90}{60} + \frac{80}{60} + \frac{72}{60}$$

their sum comes out to be equal to $\frac{163}{60}$

3. Yet in another example,[70] after reducing the fractions $\frac{12}{19}, \frac{4}{7}, \frac{6}{11}$ to a common denominator, they become :-

$$\frac{924}{1463}, \frac{836}{1463}, \frac{789}{1463}$$

And, their sum is stated to be equal to 2558.

PLAN OF WRITING EQUATIONS

In the Bakhshāli mathematics two sides of an equation are written down one after the other in the same line without any sign of equality being interposed. Thus the following:-

i.

0	5	*yu°*	*mū°*	0	*sa°*	0	7+	*mū°*	0
1	1			1		1	1		1

[71]

which means — $\sqrt{x+5} = S$, $\sqrt{x-7} = t$.

ii.

0	2	1	3	3	12	4	*dr°* 300
1	1	1	1	1	1	1	1

[72]

which means— $x + 2x + 3 \times 3x + 12 \times 4x = 300$.

In the above examples, we see that the unknown quantity has been indicated by zero (0). Sometimes the unknown quantity is not indicated e.g.–

iii.

1	1	1	*drishya* 65
2	3	4	1

[73]

which means— $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 65$.

iv.

1	3	9	*drishya* 130
1	1	1	1

[74]

which means— $x + 3x + 9x = 130$.

SQUARE ROOT

In the Manuscript, there is a rule given for determining the square-root of a non-square number. No doubt, the rule is not preserved in entirety but is partially preserved on three different folios in the Manuscript. The fragments pieced together enable us to restore the rule completely as under :

"*akṛite śliṣṭha kṛityūnān śeṣa cchedo dvi-saṁguṇaṁ.*|

tad varga dala saṁśliṣṭhaḥ hṛiti suddhi kṛiti kṣayaḥ||"[75]

Kaye[76] translates it as follows-

"The mixed surd is lessened by the square portion and the difference divided by twice that. The difference is divided by the quantity and half that squared is the loss." B.B. Datta[77] discards the translation given by Kaye. He translates the above *Sutra* in the following manner—

"In case of a non-square (number), subtract the nearest square number, divide the remainder by twice (the root of that number). Half the square of that (that is the fraction just obtained is divided by the sum of the root and the fraction (saṁśliṣṭa) and subtract; (this will be the approximate value of the root) less the square (of the last term).

Now, the symbolic representation of the *Sūtra* is—

$$\sqrt{A} = \sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{\left(\frac{r}{2a}\right)^2}{2\left(a + \frac{r}{2a}\right)}.$$

where a is the greatest root, r is the original term minus the square of the greatest root (i.e. $A - a^2$) according to the problems solved after the *sūtra* of this rule in the *Bakhshāli Manuscript*. *Mūla* is the Sanskrit form of 'root'. Since it has been used in all the examples of 'square-root', *mūlam* must be the term used for root in the Manuscript. It already occurs in the *Anuyogadvāra Sutra* (c. 100 B.C.) and in all the mathematical works.[78]

Examples of Square-Root—

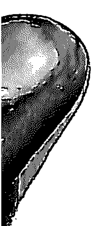
i. |1024| *asya mūlaṁ* |32|[79]

or 1024 its root is 32.

i.e. $\sqrt{1024} = 32.$

ii. **Problem :**

|889| .... *labdhaṁ mūlaṁ* |29 48 58|[80]



Solution :

$$\sqrt{889} = \sqrt{(29)^2 + 48}$$

$$\text{or } \sqrt{889} = 29 + \frac{48}{58} - \frac{\left(\frac{48}{58}\right)^2}{2\left(29 + \frac{48}{58}\right)}$$

$$\sqrt{889} = 29 + \frac{48}{58} - \frac{\left(\frac{48}{58}\right)^2}{\frac{1730}{29}}$$

$$= 29 + \frac{48}{58} - \frac{48 \times 6}{29 \times 865}$$

$$= \frac{1495874}{50170} = \frac{747937}{25085}$$

$$= 29.828$$

(iii) **Problem :**

| 481 | *mūlam śliṣ ṭha karaṇyā* | 21 | 40 | 42 | [81] ....

Solution :

$$\sqrt{481} = \sqrt{(21)^2 + 40}$$

$$= 21 + \frac{40}{42} - \frac{\left(\frac{40}{42}\right)^2}{2\left(21 + \frac{40}{42}\right)}$$

$$= 21 + \frac{40}{42} - \frac{\left(\frac{40}{42}\right)^2}{\frac{922}{21}}$$

$$= 21 + \frac{40}{42} - \frac{200}{2681} = \frac{425042}{19362}$$

$$= 21.9524.$$

(IV) **Problem**

| 41 | *mūlaṁ* | 6 | 5 | 6 | [82] ....

Solution :

$$\sqrt{41} = \sqrt{(6)^2 + 5}$$

$$\sqrt{41} = 6 + \frac{5}{12} - \frac{\left(\frac{5}{12}\right)^2}{2\left(6 + \frac{5}{12}\right)}$$

$$= 6 + \frac{5}{12} - \frac{\left(\frac{5}{12}\right)^2}{\frac{77}{6}}$$

$$= 6 + \frac{5}{12} - \frac{25}{1848} = \frac{11833}{1848}$$

$$= 6.41667.$$

RULE OF THREE —

The Rule of Three is indicated by the term *trairāśika* or three terms i.e. the rule of three terms. Bhāskara explains the term as "Here three quantities are needed (in the statement and calculation) so the method is called *Trairāśika* (the rule of the three terms)"[83]. The proposition is generally set out in the following manner in our text—

1 3	1 1 2	4 1

phalaṁ $\begin{matrix}18\\1\end{matrix}$

which means $\frac{1}{3} : 1\frac{1}{2} :: \frac{4}{1} : 18$

Here the mention of the term *phalaṁ* is of interest as it is used throughout our text, in the sense of answer and is not applied to the second term of a proportion as in the *Līlāvatī*.

(i) *udaº* || *sa .... lavanasya rāṣe koṣṭhatāṁ vā kṛitāṁ rharai | eṣaṁ chaikāṁ rāsi*

punare .... dhā nītā | saptāṇām m api chaikā rāśis tulitāṃ |

paṁcha saptatyā .... sahasraṁ bhavet saptāṣṭa guṇaṁ kiṁ

ra 1 1	1075 1	56 1

adha cchedaṁ 2000 paº bhaº | pha – bhā 30 paº

200. *eṣa rāśi lavaṇa pramāṇaṁ*[84].

Translation :

The example appears to refer to heaps of salt. If one heap or quantity weighs 1075 palas how much will 56 heaps weigh.

Solution :

$$1 : 1075\text{ P} :: 56 : x$$
$$x = 1075p \times 56$$
$$= 60200 \textit{ palas.}$$

2000 *palas* = 1 *bhāra*

$\therefore$ 60200 *palas* $= \frac{60200}{2000}$

= 30 *bhāras* and 200 *palas*

which is the answer indicated by *phalam* in the text.

(ii) *uda°* ||*dināra ko nāma viśā* .... *tti du khārjanīyaṁ sukha-bhojane cha'* | |

tasyārdhaṁ ardhaṁ cha yad ardhaṁ ardhaṁ ta ke .... *deva guru*

prasādaṁ kṛipana dhma bhuktaṁ ||

1	1	1	1	1	1	1	108
1	2	2	2	2	2	2	1

pha° dī° 1 dhā° 8 aṁ° 1[85].

Translation : The earning of dinaras is difficult but consuming them is easy. One gives one-half increased by ration of one-half (six-times) for food for the poor. What is the amount consumed in 108 days :

Solution :

$$1 : \frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2} :: 108 : x$$

$$1 : \frac{1}{64} :: 108 : x$$

$$\text{or } x = \frac{108}{64} \text{ dināras}$$

$$\text{or } x = \frac{27}{16} \text{ dināras}$$

$$= 1.6875 \text{ dināras.}$$

Now, we shall convert .6875 *dināras* into *dhānakā*, where 12 *dhānakā = 1 dināra*.. Therefore, .6875 *dināra* = 8.25 *dhānakā so, we get the value of x = 1 dinār and 8.25* dhānaka.

Since 1 *dhān akā* = 4 *aṁsa*. Therefore, .25 *dhānakās* = $\frac{25}{100} \times 4 = 1$ *aṁsa*

Hence the value of x = 1 *dināra, 8 dhānakā* and *1 aṁsa*, which is the answer indicated by *phalaṁ* in the text.

(iii) *uda°* ||..................... *vyūha pārthaṁ hehayakī* .... *ghnata*

sāyakaiś chaiva patti sva-pāda dala śodaṣai |

a .... *nya chatasrā vai hatā tena mahātma vaṁ* ||.

śarāṇāṁ cha parimāṇaṁ .... *viśārada* || .

śi 1	16	4 a° *chhe°* 21870
1	1	1 1
	1	
	4	
	1	
	1	
	2	

phalaṁ śarā 2624400 *anyā* I

pramānam[86]

Translation : This problem appears to relate to *pārtha*, the *Mahābhārata* hero, who pierced each soldier with $16\left(1+\frac{1}{4}\right)\left(1+\frac{1}{2}\right)$ arrows and slew four divisions of the army. How many arrows did he use ?

Solution :

$$1:16\left(1+\frac{1}{4}\right)\left(1+\frac{1}{2}\right)::\frac{4\text{ akshauni}}{\text{soldiers}:x}\text{or }21870$$

$$1:16\left(1+\frac{1}{4}\right)\left(1+\frac{1}{2}\right)::4\times 21870:x$$

or $$1\,\text{sP}:16\left(1+\frac{1}{4}\right)\left(1+\frac{1}{2}\right)::4\times 21870:x$$

or $$1:16\left(\frac{5}{4}\right)\left(\frac{3}{2}\right)::4\times 21870:x$$

or $$x = 87480\times\frac{16\times 5\times 3}{4\times 2}$$

or $$x = 87480\times 30$$

Hence x = 2624400, which is the answer indicated by *phalam* in the text.

(iv) *uda°* *paṁchārdha saṁvatsare bhukte rāśaikā yadi bhānujah brūhi* .... *ka tatvajña samaśve vāsareṇa kiṁ*

2 1 2	*rā°* 1	1 *aṁ°* 1 1 360

ūrdha cchedaṁ 108000 viliptānāṁ rāśi | adha cchedam 1/6 viliptā liptā ||

phalam viliptā 2 || eṣa graha gatiṁ || [87].

Translation : If *Bhānuja* (Saturn) moves through a sign in two and a half years, state, O knower of the truth, what will its motion in a solar day be equal to.

Solution : $2\frac{1}{2}$ years : 1 *rāśi* : : 1 *aṁsa* : x

Since, 1 *rāśi* = 108000 *Viliptā*

and 1 *aṁsa* = $\frac{1}{60}$ *Viliptā*

Therefore, $\frac{5}{2}$ years : 108000 *vi°* : : $\frac{1}{60}$ *vi°* : x.

Converting years into days, we get–

$$\frac{5}{2} \times 360 : 108000 :: \frac{1}{60} : x.$$

$$\text{or } \left(\frac{5}{2} \times 360\right) x = \frac{108000}{60}$$

$$\text{or } 900x = \frac{108000}{60}$$

$$\text{or } x = \frac{108000}{60} = 2 \text{ } \textit{viliptā,}$$

which is the answer indicated by *phalaṁ* in the text.

RŪPONA METHOD OR SUMMATION OF SERIES

In the *Bakhshāli* Manuscript, there are several mentions of *rūpona* method, and mention of the phrases *rūpona karaṇena*[88] or *Pratyaya rūpoṇa karaṇena.* In every case the application is to the summation of a series, in arithmetical progression. The rule[89] is –

$$S = \left[(t-1)\frac{d}{2} + a\right] t.$$

The term *rūpona* literally means 'deducting one'. *Rūpoṇa karaṇena* seems to imply that the rule in question began with the term *rūpona* which corresponds to the (t-1) of the formula, according to Kaye[90]. As the rule is not preserved in the available portion of the Manuscript, it is not possible to verify this supposition. Mahāvīra in his *Gaṇita-Sāra-Saṅgraha* (ii, 63) gives the following rule for calculating the summation of series

rūpenono gaccho dalīkṛitaḥ pra chayatādi to miśraḥ |

prabhavena padābhyas tas saṅ-kalitaṁ bhavati sarveṣām ||

which means – The number of terms is diminished by one, halved and multiplied by the increment. This when combined with the first term of the series and multiplied by the number of terms becomes the sum of all. *Rūpona* method is exemplified in two ways in our text of which the following are particular cases –

uda°

a° 1	*u°* 1	*pa°* 19
1	1	1

rūpoṇā karaṇena phalaṁ 190 / 1 [91]

which means, *ādih* (first term) = 1,
uttarah (common difference) = 1,
pada (number of terms) = 19.

Solution : The rule as we know is,

$$S = \left[(t-1)\frac{d}{2} + a \right] t$$

so, $$S = \left[(19-1)\frac{1}{2} + 1 \right] 19 = \left[\frac{18}{2} + 1 \right] 19$$

$$= (10)\ 19 \quad = 190.$$

uda

a° 3	u° 4	pa° 3
1	1	1

rūpona *karaṇena phalam rū°* 21|| [92]

Solution :

$$S = \left[(3-1)\frac{4}{2} + 3 \right] 3$$

$$= 7 \times 3 \quad = 21$$

Problems regarding toll-duties —

uda|| *datvā śulkaṁ chatur bhāgaṁ*
aṣṭau āṇīta kuṁkumā |
chatu śulka śālais tu kiṁ śesaṁ
vada paṇḍita ||

8	1
1	1
	4 +

karanaṁ | *kritvā rūpa kṣayaṁ pāstaṁ pāstaṁ*

8	3
1	4

guṇitaṁ jātaṁ $\boxed{6}$ *śulke*

2 *śeṣaṁ* 6/1   1/1/4 +   *anena guṇitaṁ*

jātaṁ	4	*kṣyaṁ*	1	*śeṣeṇa*	4	1
	1		1		1	1
	2		2		2	4 +

datvā guṇita jātā $\boxed{\begin{matrix}27\\8\end{matrix}}$ 93 ....

Translation : Having given one-quarter as toll at four toll-houses eight of saffron is brought in. State, O Pandit, what is left.

Solution : $8 \times \frac{3}{4} = 6$ and 2 is paid in toll;

$6\left(1-\frac{1}{4}\right) = 4\frac{1}{2}$

The loss is $1\frac{1}{2}$

$4\frac{1}{2}\left(1-\frac{1}{4}\right) = \frac{27}{8} = 3\frac{3}{8}$

The toll = $1\frac{1}{8}$

$3\frac{3}{8}\left(1-\frac{1}{4}\right) = \frac{81}{32}$

The last toll = $\frac{27}{32}$

Total toll paid = $2 + 1\frac{1}{2} + 1\frac{1}{6} + \frac{27}{32}$

$= 5$

which leaves $8-5\frac{15}{32} = 2\frac{17}{32} = \frac{51}{32}$.

Sutram||*Vastra śulkam yad bhavati tada ...... hṛita vastratam̐* |

trai-rāśika vidhānena śulka vikraya tatvataḥ || 94.

Translation: (Rule) That which is the tax on cloth, by the method of rule of three and sale alike.

The text of the example of this *Sūtra* is very fragmentary and hence uninteligible. The solution is missing.

uda° |*mākshikag - ghaṭakasyaiva dvi-tri-bhāga pravardhitam̐* |

dvitiye dvi-pam̐chamo-bhāgo tritīye dvi-sapatakodbhavam̐ |

chaturthe dvi-navam̐-bhāgam̐ evam̐ jāta pala trayam̐ |

babhūvā śaulkikai hṛitvā kim̐ sarvam̐ vada paṇḍita ||

2 3	2 5	2 7	2 9	se° 3 1

dhāntaso .... iti | 95.

Translation : Of a *ghaṭaka* of honey two-thirds is given, to the second two-fifths, to the third two-sevenths, to the fourth two-ninths, till only three *palas* (are left). O Pandit, state how much altogether was taken away by the tax-collector.
The solution is missing in the available text.

Problems Regarding 'Profit And Loss'

uda° |*Chatu Pam̐chaka lābhena daśa droṇāt prayojita*|

tad vai tribhis tu kim̐ lābham katthyatam̐ gaṇakottama ||

10 1	5 4	5 4	5 4	*guṇitam jātam*	1250 64

96.

Translation By a gain of five-fourths ten *dronas* are obtained. Let it be said, O best of calculators, what will be the gain by three transactions.

Solution : $10.\frac{5}{4}.\frac{5}{4}.\frac{5}{4} = \frac{1250}{64} = \frac{625}{32}$ *dronas*

= *19 Droṇa, 2 āḍhaka, o prasthas and 2 kuḍavas*, where —

1 *Drona* = 4 *aḍhakas*
1 *aḍhakas* = 4 *prasthasa*
1 *prastha* = 4 *kuḍavas*

udaº || *kasyāpyarjakasya ṣ aṣ ṭhisva-dalena kṣayaṁ gata* |

puna vṛiddhyā tri-bhāgena sva-pādena tatojjhitaṁ

vriddhyā tu paṁcha-bhāgenas tathā vṛiddhi dvayo gataṁ |

kā vṛiddhi .... *syā kiṁ vā śeṣaṁ tad uchyatāṁ* ||

60	1	1	1	1
1	1	1	1	1
	2+	3	4+	5

rūpa lā ...... *jātā* 36||

pratyayaṁ punasyaiva

0	1	1	1	1		36
1	1	1	1	1	*bhaº*	
	2+	3	4+	5		1

phalaṁ 60||

punānyaṁ pratyayaṁ 60 1 1 2+ 1 3 1 4+ 1 5 *phalaṁ* 36|| ....

.... *mulaṁ na jñayate* 0 1 1 2+ 1 3 1 4+ 1 5 *phalaṁ*.........97.

Translation The capital of a certain banker is sixty. One half of it goes in loss and then he gains by one-third; next he loses one-fourth of it and finally gains one fifth; so that he has two gains. What is his gain and what is his loss and the remainder and let that be stated.

Solution : $60\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right) = 36.$

Proof : (a) $x' = \dfrac{36}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)}$, whence $x' = 60$.

(b) $60\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right) = 36.$

(c) $x'\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right) = 36,$

whence $x' = 60$.

udaº ||*ajñātārambha - lohasya trichatu paṁchakā kṣaye* |
sapta-viṁśati piṇḍasya tridhānta śeṣya dṛiṣyate |
kiṁ sarvaṁ vada tatvajña kṣayaṁ cha mama katthyatāṁ ||

1/3	1/4	1/5	śeº 27/1

karaṇaṁ | *kritvā rūpa kṣ· ayam pāstha*

2/3	3/4	4/5	*guṇitaṁ jātaṁ*	2/5

rūpa kṣ, ayaṁ | 3/5 | *anena śeṣaṁ*

bhaktaṁ śeṣaṁ | 27 | *bhaktaṁ jātam 45 asya saptā-viṁśa* | *pātya śeṣam 18* | *eta kṣ ayam* ||[98].

Translation : An unknown quantity of lapislazuli loses one-third, one-fourth, and one-fifth; and the remainder after the three-fold operation on the original quantity is twenty-seven. State what the total was, O wise man, and also tell me the loss.

Solution : $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}=\frac{2}{5};\ 1-\frac{2}{5}=\frac{3}{5};$

$27 + \frac{3}{5} = 45$

$\therefore\ 45 - 27 = 18$, this is the loss.

sutram || *Vikrayena krayam bhājyaṁ rūpa hīnaṁ punar bhājet*. ||
lābhena gunaye tatra nīvī bhavati tatra cha ||[99].

Translation : The rule means $C = \dfrac{P}{c/s - 1}$.

where C is the capital, P the profit, c the rate of purchase and s the rate of sale.
The following *udhāharaṇa* (example) has been given with the above-mentioned *Sūtra* (Rule).

udā° || dvibhi krināti yas sapta vikrināti tribhiṣa śat |
aṣṭā – daśa bhaved lābha kā nīvī tatra katthyatām ||

7	6	18	labha
2	3	1	

karaṇam || vi .... nivi jātā | sya pratyaya trairāśikena ||
yadi dvibhis sapta labhyate | tadā chaturvimśatibhi kim |

2	7	24
1	1	1

phalaṁ rū° 84 ||

asya vikrayaṁ kriyate |yadi saḍbhi traya .... labhyate tadā chaturāśitibhi kiṁ |

6	3	84
1	1	1

phalaṁ 42 | mūlaṁ 24 |

pātya śeṣam 18 eśa lābhāḥ[100].

Translation : One buys 7 for 2 and sells 6 for 3 and 18 is his profit. What was his capital ?

Solution :

$$C = \frac{P}{C/_{s} - 1} = \frac{18}{\dfrac{7/2}{6/3} - 1}$$

where P = 18, C = 7/2, s = 6/3

so, $C = \dfrac{18}{\frac{3}{4}} = 24$

Now, If for 2, 7 are obtained, then what for 24.

2 : 7 : : 24 : x articles

$2x = 24 \times 7$

or $x = \dfrac{24 \times 7}{2} = 84$ articles

Again, If by 6, 3 are obtained, then what for eighty.
6 : 3 : : 84 : x

or $6x = 84 \times 3$

or $x = \dfrac{84 \times 3}{6} = 42$

Hence, the original quantity = 24
∴ the difference = 42 – 24 = 18.

Sutram | *Vikrayaṁ bhājaye chaiva guṇa - yet kraya piṇḍatāṁ* |
rūpone mūla guṇye labdhā lābhaṁ cha prāpyate || [101].

Translation : The rule means — $P = C\left(c/s - 1\right)$

where P is profit, C is the capital, c is the rate of purchase and s is the rate of sale.

The following *udāharaṇa* (example) has been given with the above-mentioned *Sūtra* (Rule).

uda° || *dvibhi krīṇati yas sapta vikriṇāti tribhiṣa ṣat* ....
mūla chā ....

2	7	24	pha° 84
1	1	1	1

atha vikrayaṁ

6	3	84	pha° 42
1	1	1	1

.....................24

pātya śeṣaṁ 18 | *eṣa lābhaṁ* || [102].

Examples : Articles are bought at 7 for 2 and sold at 6 for 3.

Solution : We know P = C (c/s − 1)

Here C = 24, $c = \frac{7}{2}$, $s = \frac{6}{3}$

$$\therefore\ P = 24\left(\frac{7}{2} \div \frac{6}{3} - 1\right)$$

$$= 24\left(\frac{9}{12}\right) = 18.$$

So, profit = 18.

Proof : 2 : 7 : : 24 : X

2x = 24 × 7

or $x = \frac{24 \times 7}{2} = 84.$

Hence 2 : 7 : : 24 : 84.

and 6 : 3 : : 84 : x

6x = 84 × 3 or $x = \frac{84 \times 3}{6} = 42$

Hence 6 : 3 : : 84 : 42

Therefore 42 − 24 = 18 is the profit.

Problems Regarding 'Ratio And Proportion'

Sutra || *Yadricchā pinyase śūnye tadā vargaṁ tu kārayet* ||[103].

Translation : "Put into the empty place the number 1, representing the desired quantity, and then make up the series of items."

The second portion of this *Sutra* is not preserved.

The following *udāharaṇs* (examples) will help us in uderstanding this rule completely—

uda° || .... ||*tadā cha triguṇam dattaṁ* .... || .... *prathamasya tu kiṁ bhavet* ||

0	*tadā* 2	*tadā* 3	*tadā* 4	*dattaṁ* 132
1	1	1	1	1

Karaṇaṁ | *yadṛcchā vinyase śūnye* | *tatreccha* | 1|

tadā vargaṁ tu kārayet

1	2	3 2	6 4
1	1	1 1	1 1

prakṣipe guṇitaṁ 1| 2| 6| 24|.................................

prakṣiptaṁ 33|| *dṛiṣyaṁ vibhajet* $\frac{132}{33}$

vartyam jātam $\frac{4}{1}$ eṣa prathameṇa dattaṁ || ato nyāsaḥ||

4 |8 | 24 | 96 | dattaṁ 132 eṣa vargakramagaṇitam ||[104].

Example : B gives 2 times as much as A, C gives 3 times as much as B, D gives 4 times as much as C. Their total gift is 132. What is the gift of A.

Statement : A Gives x, B 2, C 3, D 4. Total 132.

Solution : "Put 1 in the place of x.
Now form the series of items = 1, 2, 3 × 2, 4 × 6,
Multiplying these several rates, we get 1, 2, 6, 24.
Total = 33

Dividing given total 132 by 33 = $\frac{132}{33} = 4$

∴ The result item = 4.

Therefore, the gift of A = 4.
Hence the series of gifts = 4, 8, 24, 96
∴ The total gifts = 132, as is given also.
This is calculated by the series of items.

Sutram ||*Kāmikam Śūnyavinyastaṁ tadā chaiva krame guṇaṁ* |

udā° || .... |
......................*Kṛitvā chaturtha*.. |
.... *Prathamasyatu kiṁ bhavet* ||

sthāpnaṁ

0	2	1	3	3	12	4	dṛi°	300
1	1	1	1	1	1	1		1

kāmikaṁ śūnye pinyastaṁ kāmikaṁ 1|| *eṣa nyāstam prathamarāsau* | *tadā chaiva kramena guṇitam* | 1| 2| 9| 48| *eṣām yuti prakṣepaṁ* $\frac{60}{1}$ *anena dṛiśyaṁ bhājitaṁ* $\frac{1}{60}\ \frac{300}{1}$ *jātā* |5| *eṣa prathamasya dhanaṁ* | *anena kṣepam guṇaye* |5| 10| 45| 240| *evaṁ 300 eṣa yutivargagaṇitaṁ* ||[105].

Translation The above mentioned line before the example seems to be the modification of the same *Sutra* and will again help us in understanding the *Sūtra*. Since it is not specialised as a separate sūtra, what remains of it, runs like this – 'the number 1 is put into the empty place, and then (the items) are successively multiplied.

Example B possesses 2 times as much as A; C has 3 times as much as A and B together; D has 4 times as much as A, B and C together. Their total possessions are 300. What is the possession of A.'

Statement A has x, B = 2, C = 3 × 3, D = 4 × 12. Total = 300

Solution The desired quantity is put in the empty space.
By putting desired quantity 1 as the first number, the successive muliplications are 1, 2, 9, 48.

Sum of rates = 1 + 2 + 9 + 48 = 60.

Dividing given total 300 by 60 = $\frac{300}{60}$ = 5.

∴ Possession of A = 5.

So, the several rates are = 5, 10, 45, 240.
Hence, total of the items = 300.

udā° || *Prathamasya na jānāmi kathaṁ dattaṁ chaivā dhanaṁ* |
Sa cha dvyārdha yutaṁ dhanaṁ.. || [106].

Translation This fragment of third example seems to be a third modification of the same *Sūtra*, which is lost.

udā° || .... |
.... *śataṁ chatuś-chatvālimśādhikaṁ* |
kiṁ prathamasya dhana .... ||

0	1	2	2	3	3	4	4	dr̥	144	....
1	1	1	1	1	1	1	1		1	
	2		2		2		2		2	

.... *śūnyesu* $\begin{array}{|cc|}1 & 1\\ 1 & 1\\ & 2\end{array}$ *yutam chaiva guṇam tataḥ |*
yutam chaiva guṇam kṛitvā kāraye gaṇakramantu $\begin{array}{|c|}5\\ 2\end{array}$
guṇaṁ | upare uparaṁ adhe adhaṁ guṇaye $\begin{array}{|c|}10\\ 2\end{array}$
sārdhadvayayutaṁ $\begin{array}{|c|}15\\ 2\end{array}$ *tritiyarāśyaguṇanaṁ | sārdhais saptabhi triṇi* $\begin{array}{|c|}45\\ 2\end{array}$ *Sārdhatrayayutaṁ* $\begin{array}{|c|}52\\ 2\end{array}$ *chaturtharāśi guṇaye-ṣaḍviṁśatibhi |*
jāta $\begin{array}{|c|}208\\ 2\end{array}$ *sārdhachatvāriyutaṁ* $\begin{array}{|c|}217\\ 2\end{array}$ *prakṣepayuti* $\begin{array}{|c|}289\\ 2\end{array}$
evaṁ drisyam | sarvaṁ tadeva jātam || [107].

Example A possesses something and $1\frac{1}{2}$ in addition; B has 2 times as much as A and $2\frac{1}{2}$ in addition; C has 3 times as much as B and $3\frac{1}{2}$ in addition; D has 4 times as much as C and $4\frac{1}{2}$ in addition. Their total possessions $= 144\frac{1}{2}$. We have to find the possession of A.

Statement

$A = x + 1\frac{1}{2}$, $B = 2 + 2\frac{1}{2}$,

$C = 3 + 3\frac{1}{2}$, $D = 4 + 4\frac{1}{2}$.

$$\text{Total} = 144\frac{1}{2}.$$

Solution

Put 1 in the empty place, we get $1 + 1\frac{1}{2}$.

"The several additions and multiplications are then made." In making the additions and multiplications, let the proper order of calculation be observed.

By addition $\frac{5}{2}$, next comes multiplication; i.e. multiply numerator with numerator and denominator with denominator.

$$\text{i.e. } \frac{2}{1} \times \frac{5}{2} = \frac{10}{2}$$

$$\text{Adding } 2\frac{1}{2} = \frac{10}{2} + \frac{5}{2} = \frac{15}{2} = 7\frac{1}{2}$$

Multiplying it with third number or three, we get.

$$\frac{15}{2} \times 3 = \frac{45}{2}$$

Adding $3\frac{1}{2}$, we get $= \frac{45}{2} + 3\frac{1}{2} = \frac{45}{2} + \frac{7}{2} = \frac{52}{2} = 26.$

Multiplying 26 by 4 $= 104 = \frac{208}{2}$

Adding $4\frac{1}{2}$ or $\frac{9}{2} = \frac{208}{2} + \frac{9}{2} = \frac{217}{2}.$

Since the total of these rates $= \frac{289}{2}$, which is given.

Sum of rates, we get $= \frac{289}{2}$, i.e. $\frac{5}{2} + \frac{15}{2} + \frac{52}{2} = \frac{289}{2}.$

Dividing given total by the sum of the rates,

we obtain $\frac{289}{2} \div \frac{289}{2} = 1$.

So, $x = 1$

Hence the possessions of A, B, C, D are respectively

$\frac{5}{2}$, $\frac{15}{2}$, $\frac{52}{2}$ and $\frac{217}{2}$.

The above mentioned problem can also be solved as follows —
Putting x in the empty place, we get —

$A = \left(x + 1\frac{1}{2}\right)$, $B = 2\left(x + 1\frac{1}{2}\right) + 2\frac{1}{2}$,

$C = 3\left[2\left(x + 1\frac{1}{2}\right) + 2\frac{1}{2}\right] + 3\frac{1}{2}$,

$D = 4\left[3\left\{2\left(x + 1\frac{1}{2}\right) + 2\frac{1}{2}\right\} + 3\frac{1}{2}\right] + 4\frac{1}{2}$

It is givent that $A + B + C + D = 144\frac{1}{2}$.
Substituting values for A, B, C, D, we get —

$$x + \frac{3}{2} + 2x + 3 + \frac{5}{2} + 6x + 20 + 24x + 80 + \frac{9}{2} = \frac{289}{2}$$

or $33x + \frac{223}{2} = \frac{289}{2}$.

or $33x = \frac{289}{2} - \frac{223}{2} = \frac{66}{2} = 33$.

$\therefore\ x = 1.$

Hence the possessions of A, B, C, D are respectively

$\frac{5}{2}$, $\frac{15}{2}$, $\frac{52}{2}$ and $\frac{217}{2}$.

i.e. by substituting the value of x, we get these values.

uda° || .... |
.... *triguṇam tṛi sārdhayutaṁ* || *Chaturguṇam chaturthena navārdhayutaṁ dattaṁ* | .... *dvisata dvavimsadhika* || *kim atra prathamasya dattāsit*

0 / 1	3 / 2	2 / 1	5 / 2	3 / 1	7 / 2	4 / 1	9 / 2	*ekatram dattam* 222

Śūnya sthāne rūpam datvā | 1 | *yutaguṇita yutakrameṇa jātam* |

Sthāpanaṁ–

$\frac{5}{2}$	$\frac{15}{2}$	$\frac{67}{2}$	$\frac{357}{2}$	*dṛiśya* 222	*prakṣepena jātam* 222

dattaḥ driṣyah 222||[108].

Translation : **Example** – A gives 3/2 plus a certain amount; B gives 5/2 plus 2 times as much as A, C gives 7/2 plus 3 times as much as A and B; D gives 9/2 plus 4 times as much as A, B, C. The total of their gifts is 222. What was the gift of A ?

Statement : A gives $x + \frac{3}{2}$, $B = 2 + \frac{5}{2}$, $C = 3 + \frac{7}{2}$, $D = 4 + \frac{9}{2}$; the joint gift is 222.

Solution : "Having put the number one in the empty place; 1 (for x), the additions and multiplications are made in their proper order. The result is the following series of rates : $\frac{5}{2}, \frac{15}{2}, \frac{67}{2}, \frac{357}{2}$; the given total is 222. The addition of the rates yields 222, which is same as the given total. This practically finishes the solution.

Let us solve the above examples as follows-

By putting x in the empty place, we get

$A = \frac{3}{2} + x$, $B = \frac{5}{2} + 2\left(\frac{3}{2} + x\right)$, $C = \frac{7}{2} + 3\left[\left(\frac{3}{2} + x\right) + \frac{5}{2} + 2\left(\frac{3}{2} + x\right)\right]$

and $D = \frac{9}{2} + 4\left[\left(\frac{3}{2} + x\right) + \frac{5}{2} + 2\left(\frac{3}{2} + x\right) + \frac{7}{2} + 3\left\{\left(\frac{3}{2} + x\right) + \frac{5}{2} + 2\left(\frac{3}{2} + x\right)\right\}\right]$.

It is given that $A + B + C + D = 222$.

i.e. $\frac{3}{2} + x + \frac{5}{2} + 3 + 2x + 9x + \frac{49}{2} + 48x + 126 + \frac{9}{2} = 222$

or $60x + 162 = 222$.

or $x = \frac{222 - 162}{60} = \frac{60}{60} = 1$

By substituting the value of x, we get,

$$A = \frac{5}{2},\ B = \frac{15}{2},\ C = \frac{67}{2} \text{ and } D = \frac{357}{2}$$

Problems pertaining to Wastage of Gold

Idānim suvarṇa kṣayam vakṣyāmi ....syedam

Sutram|| Kṣayam Saṁguṇya kanakās tadyutir bhājayet tatah
saṁyutair eva kanakair ekaikasya kṣayo hi sah |[109].

Translation : The line above the sutra means "Now I shall discuss the wastage (in the working) of gold, the rule about which is as follows :

Rule : Having multiplied severally the parts of gold with the wastage, let the total wastage be divided by the sum of the parts of gold. The result is the wastage of each part (of the whole mass) of gold.

udaº || eka dvitri chatus saṁkhyā suvarṇā māsakai rinai|
eka dvitṛi chatus saṁkhyā rahitā sama bhāgataṁ ||
sthāpanaṁ kriyate | eṣāṁ

1 +	2 +	3 +	4 +
1	2	3	4

karaṇaṁ|| kṣayam saṁguṇya kanakādibhi kṣayena saṁguṇya jātam

| 1| 4 | 9 | 16 | .... | *esha yuti* 30 *kanakā yuti 10*

anena bhaktvā labdhaṁ ....

1	1		
10	30	4	*phaº maº seº*12
1	1	1	1

[110]

Example *Suvaraṇas* numbering respectively one, two, three, four, are subject to a wastage of *masakas* numbering respectively one, two, three, four. Irrespective of such wastage they suffer an equal distribution of wastage (what is the latter ?)

Statement
Wastage — 1, 2, 3, 4 *maṣaka.*
Gold — 1, 2, 3, 4 *suvarṇa.*

Solution Having multiplied severally the parts of gold with the wastage, etc; by multiplying with the wastage, the products 1, 4, 9, 16 are obtained; "let the total wastage", its sum is 30; the sum of the parts of gold is 10; dividing with it, we obtain 3. (This is the wastage of each part, or the average wastage, of the whole mass of gold.)

Proof :- By the rule of three, we get —
10 : 30 : : 4 : x
10x = 120
x = 12

$\therefore$ the sum of gold = 10.
Wastage on 10 = 30 *masakas*.
and the sum of gold = 4
wastage on 4 = 12 *masakas*.

udaº *eka dvi tri chatu saṁkhyā suvarṇa projjhitā ime māśaka dvi tritāṁ chaiva chatu saṁkhyā paṁchakarāṁśakaṁ kim kṣayam*

1	2	3	4
1	1	1	1
2	3	4	5

karaṇaṁ || *kṣayam saṁguṇya kanakā eṣa sthāpyate* |

1	2	3	4
2	3	4	5

tad yutir bhājayet tatah hara sāsye kṛite yutaṁ

163
60

saṁ-yutai kanakair bhaktvā tadā kanaka $\boxed{10}$

anena bhaktam jātam $\frac{163}{600}$ *eṣa ekaika suvarṇasya kṣayam* ||

pratyaya trai - rāśikena....

10	163	1	*phaº*	163
1	60	1		600
10	163	2	*phaº*	163
1	60	1		300
10	163	3	*phaº*	163
1	60	1		200
10	163	4	*phaº*	163
1	60	1		150

|||

Example There are *Suvaraṇas* numbering one, two, three, four. There are thrown out the following *maśakas*; one-half, one-third, one-fourth, one-fifth. What is the (average) wastage (in the whole mass of gold) ?

Statement Quantities of gold, 1, 2, 3, 4, *suvarṇa*

Wastage $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ *maśaka*

Solution Having multiplied severally the parts of gold with the wastage, etc.;

the products may thus be stated - $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

Let the total wastage be divided; the division being directed to be made, the total wastage is $\frac{163}{60}$;

dividing by the sum of the parts of gold, here the sum of the parts of gold is 10. being divided by this, the result is $\frac{163}{600}$.

$\therefore$ $\frac{163}{60}$ is the wastage of each part of the whole mass of gold

Proof By rule of three :
the sum of the parts of gold, 10 : the total wastage of $\frac{163}{60}$ maśaka.
so, the sum of gold 4 : the wastage of $\frac{163}{150}$ maśaka, etc.

PROOFS

Proofs are given at the end of many solutions of the problems. The usual term used in the text to denote 'proof' or 'verification' is *pratyayam*. The proofs of the problems, dealing with 'rule of three' are indicated by *pratyaya – trai-rāśikena*. 'Proof by the rule of three'. Similarly 'Proofs' to the solutions of the problems dealing with the *rūpona* method are indicated by Pratyaya–rūpona–karaṇena 'Proof by the rūpona method' some of the specimens of the proof are given below –

(1) Problem $\frac{5}{3}t - 7 = \frac{6}{5}t + 7$

Solution $t = 2 \times \dfrac{7}{\frac{5}{3} - \frac{6}{5}} = 30$

Proof 3 : 5 : : 30 : 50, 5 : 6 : : 30 : 36 and
50 – 7 = 36 + 7.

(2) Problem : If 7 are bought for 2 and 6 sold for 3, and the capital is 24, what will be the profit ?

Solution : $P = C\left(\frac{c}{s} - 1\right) = 24\left(\frac{7}{2} \div \frac{6}{3} - 1\right) = 18.$

Proof 2 : 7 : : 27 : 84 (the number of articles)
and 6 : 3 : : 84 : 42 (the total proceeds),
and 42 – 24 = 18.

or 1 : c : : C : n, s : 1 : : n : C + p and C + p – C = P

(3) Problem : $Dt = \frac{(t-1)d}{(2+a)t}$

Solution : $t = \frac{2(D-a)}{d+1}$

Proof by the rupona method :

$$S = \left|(t-1)\frac{d}{2} + a\right| t \text{ and } Dt = S.$$

(4) Problem : $x\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right) = x - 280.$

Solution : $x = \frac{280}{1-\frac{3}{10}} = 400.$

Proof : $\frac{400}{2} = 200,\ 400 - 200 = 200,$

$\frac{200}{4} = 50,\ 200 - 50 = 150,$

$\frac{150}{5} = 30,\ 150 - 30 = 120,$

and $400 - 120 = 280.$

REFERENCES

1. *Kaye, Bakh. Ms.* p. 15.
2. *Bakh. Ms*. folio 51 recto, p. 163.
3. *Ibid*
4. *Bakh. Ms*. folio 22 verso, p. 193.
5. *Bakh. Ms*. Ibid.
6. *Bakh Ms,* folio 1 verso, p. 169.
7. *Bakh Ms*. folio 5 recto, p. 177.
8. *Bakh Ms*. folio 15 recto, p. 207.
9. *Bakh. Ms*. folio 1 verso, p. 168.
10. *Bakh Ms*. folio 2 recto, p. 169.
11. *Ibid.*
12. *Bakh. Ms*. folio 60 verso, p. 185.
13. *Ibid*
14. *Bakh Ms*. folio 4 recto, p. 175.
15. *Ibid.*
16. *Bakh Ms*. folio 5 recto, p. 177.
17. *Bakh Ms*. folio 7 verso, p. 174.
18. *Bakh Ms*. folio 51 recto, p. 163.
19. *Bakh Ms*. folio 31 recto, p. 186.
20. *Bakh. Ms*. folio 8 recto, p. 172
21. *Bakh. Ms*. folio 9 verso, p. 174.
22. *Bakh. Ms*. folio 5 verso, p. 177.
23. *Bakh. Ms*. folio 11 vecto, p. 200.

24. *Bakh. Ms.* folio, 13 recto, p. 203.

25. *Bakh. Ms.* folio 12 verso, p. 202.

26. *Bakh. Ms.* folio 27 recto, p. 167.

27. *Bakh Ms.* folio 5 verso, p. 177.

28. *Bakh Ms.* folio 13 recto, p. 203.

29. *Bakh. Ms.* folio 65 verso, pp.179-180.

30. *Bakh. Ms.* folio 64 recto, p. 180.

31. *Bakh. Ms.* folio 3 verso, p. 171.

32. *Ibid*

33. *Bakh. Ms.* folio 38 verso, p. 61.

34. *Ibid.*

35. *Bakh. Ms.* folio 45 recto, p. 182.

36. *Bakh. Ms.* folio 38 recto, p. 161

37. Ibid

38. *Bakh. Ms.* folio 22 recto, p. 192.

39. *Ibid.*

40. *Bakh. Ms.* folio 23 verso, p. 194.

41. *Bakh. Ms.* folio 45 verso, p. 182.

42. *Bakh. Ms.* folio 10 verso, p. 199.

43. *Ibid*

44. *Bakh. Ms.* folio 10 recto, p. 198.

45. *Bakh. Ms.* folio 60 verso, p. 217.

46. *Bakh. Ms.* folio 4 recto, p. 175.

47. *Bakh. Ms.* folio 58 recto, p. 190.

48. *Bakh. Ms.* folio 22 verso, p. 193.

49. *Bakh. Ms.* folio 23 recto, p. 193.

50. *Bakh. Ms.* folio 23 verso, p. 194.

51. *Bakh. Ms.* folio 25 verso, p. 196.

52. Datta, *BCMS,* vol XXI, p. 23.

53. Hoernle, *IA* , vol. XVII, p. 30.

54. *kaye, Bakh. MS.* pp 17, 25.

55. Datta, *BCMS,* vol. XXI, p. 23.

56. *Bakh. Ms.* folio 59 recto, p. 215.

57. *Bakh. Ms.* folio 27 verso, p. 167.

58. *Bakh. Ms.* folio 9 recto, p. 173.

59. *Bakh. Ms.* folio 1 verso, p. 168.

60. *Bakh. Ms.* folio 5 recto, p. 177.

61. *Bakh. Ms.* folio 14 recto, p. 205.

62. *Bakh. Ms.* folio 56 recto, p. 180.

63. *Bakh. Ms.* folio 9 verso, p. 174.

64. *Bakh. Ms.* folio 5 verso, p. 177.

65. *Bakh. Ms.* folio 22 recto, p. 192.

66. *Bakh. Ms.* folio 23 verso, p. 194.

67. *Bakh. Ms.* folio 56 verso, p. 180.

68. *Bakh. Ms.* folio 1 verso, p. 168.

69. *Bakh. Ms.* folio 17 recto, pp. 209-210.

70. *Bakh. Ms.* folio 2 verso, p. 170.

71. *Bakh. Ms.* folio 59 recto, p. 215.

72. *Bakh. Ms.* folio 23 verso, p. 194.
73. *Bakh. Ms.* folio 70 recto, p. 185.
74. *Bakh. Ms.* folio 69 verso, p. 185.
75. *Bakh. Ms.* folios 56 recto, 57 verso; pp. 180-181.
76. *Bakh. Ms.* kaye, p. 30.
77. Datta, *BCMS*, vol. XXI, p. 11.
78. Datta & Singh,*History of Hindu Mathematics*, pp. 169-170.
79. *Bakh. Ms.* folio 5 verso, p. 177.
80. *Bakh. Ms.* folio 6 verso, p. 178.
81. *Bakh. Ms.* folio 56 recto, pp. 179-180.
82. *Bakh. Ms.* folio 56 verso, p. 181.
83. In Bhāskara's commentary on the *Āryabhaṭiya*.
84. *Bakh. Ms.* folio 36 recto, p. 225.
85. *Bakh. Ms.* folio 33 recto, p. 223.
86. *Bakh. Ms.* folio 47 verso, p. 228.
87. *Bakh. Ms.* folio 37 verso, p. 227.
88. *Bakh. Ms.* folio 9 recto, p. 173.
89. *Bakh. Ms.* folio 8 recto, p. 172.
90. *Bakh. Ms.* kaye, p. 33.
91. *Bakh. Ms.* folio 9 recto, p. 173.
92. *Bakh. Ms.* folio 7 verso, p. 174.
93. *Bakh. Ms.* folio 12 verso, p. 202.
94. *Bakh. Ms.* folio 63 verso, p. 221.
95. *Bakh. Ms.* folio 16 recto, p. 208.

96. *Bakh. Ms.* folio 13 recto, p. 203.

97. *Bakh. Ms.* folio 13 verso, p. 204.

98. *Bakh. Ms.* folio 14 verso, pp. 205-206.

99. *Bakh. Ms.* folio 61 verso, p. 218.

100. *Bakh. Ms.* folios 61 verso, 62 recto; pp. 218-219.

101. *Bakh. Ms.* folio 62 recto, p. 219.

102. *Bakh. Ms.* folio 62 verso and recto, p. 219.

103. *Bakh. Ms.* folio 22 verso, p. 193.

104. *Bakh. Ms.* folio 23 recto pp. 193-194.

105. *Bakh. Ms.* folio 23 verso, p. 194.

106. *Ibid.*

107. *Bakh. Ms.* folio 24 recto, pp. 194-195.

108. *Bakh. Ms.* folio 24 verso, p. 195.

109. *Bakh. Ms.* folio 16 verso, p. 209.

110. *Bakh. Ms.* folio 16 verso and 17 recto, p. 209.

111. *Bakh. Ms.* folios 17 recto and verso, pp. 209-210.

CHAPTER VI

i. THE SOCIO-ECONOMIC CONTENT
ii. POLITICAL THEORY AND ADMINISTRATION

VI

I. SOCIAL LIFE

The *Bakhshāli Manuscript* has not been preserved in entirety. It has come down to us in a very fragmentary condition. Only a small number of leaves are preserved, as such, the light thrown by the preserved portion on the contemporary social life is very meagre. It is not possible even to piece together, the different scraps of information, so as to present even some gleanings of the contemporary society. We give below the data, that is available to us.

1. VARṆA SYSTEM

The Hindu Social organisation is based on the institution of the *Varṇa* (the classes or the castes) and *Āśramas* (the four orders or stages of life). The information gleaned from our texts, as regards this important institution, is negligible. We find mention of only one supermost caste, i.e, *Brāhmaṇ* [1]. A Brāhmaṇ (born from the mouth of Purusha) is understood to be a theologian, a man of the first Hindu tribe[2]. His duties, according to our text are as good as, duties mentioned by *Kautilya* in *Arthaśāstra*. *Kautilya* [3] enumerates the duties of the *Brāhmaṇ* as -

1. *Adhyayana* (study)
2. *Adhyāpana* (teaching)
3. *Yajana* (worship)
4. *Yājana* (officiating at worship)
5. *Dāna* (making gifts) and
6. *Pratigraha* (accepting gifts)

The reverence with which *Brāhmaṇs*, are mentioned in our text shows that they occupied an important place and position in the social higherarchy. They were the custodians of knowledge and knowledgeable amongst them are addressed as *Paṇdits* [4]. *Paṇdit* is said to be a scholar, a teacher, a learned *Brāhmaṇ* or one well versed in sacred science, and teaching it to his disciples[5]. Different problems, both arithmetical and algebraic are addressed to *Paṇdits* for solution[6].

Brāhmaṇs also appear to have been engaged for the performance of different rituals by the house-holders and were regularly invited to feasts and festive-occasions. Inviting a *Brāhmaṇ* to a meal was considered a means for earning spiritual merit for the next world[7].

Brāhmaṇs were also the repositories of astronomical and astrological knowledge and proficient among them in these sciences were known as *Ganakottama* [8]. *Ganakott ama* a word in itself means an astrologer, a calculator of nativities, an arithmetician[9].

Even the composition of our text is attributed to a *Brāhmaṇ* whose name unfortunately, has not been preserved[10]. In our text, *Brāhmaṇ* is also mentioned as *Vipra* [11]. *Vipra* is said to be a *Brāhmaṇa*, a sage, a wise man[12]. It is true that we have no explicit mention of the other three castes. But this does not warrant the view of some scholars that in early times, there was no such caste as *Kṣatrīya*, *Vaiśya* or *Śūdra'* [13]. The author of the *Bakhshāli Manuscript* refers to the '*Rājaputras*' as the servants of king[14]. Also, *Rājaputra* means a *rājput*, who claims descent from ancient *kṣatriyas* [15]. The *Rājaputra*, here in this sense, evidently denotes the persons, belonging to the warrior caste. They were in the regular employ of kings and received wages in lieu of the services rendered.

The fact, it seems to be that as in other parts of the country so in *Gandhāra*, the system of customery four castes was very much in vogue.

2. OCCUPATIONS

Regarding the occupations pursued by people we have information of the following occupations.

The imparting of knowledge including that pertaining to mathematics, astrology and astronomy was pursued by knowledgeable persons called *Paṇdits* and *Gaṇakas*. At the same time, problems were often addressed also to an individual or individuals variously described as *jñeyah* (learned man)[16], *Tatvajña* (wise man)[17], *Dharmjña* (expert)[18], etc.

The trade was carried on by the people called *Vaṇik* [19]. *Vaṇik* generally means a merchant, a trader[20]. There is a mention of merchants dealing with the *maṇi* [21]. '*Maṇi*' means a gem, a jewel, a precious stone[22]. We learn of gifts of precious gems made by the tributory princes to *Yudhistra* on the occasion of the *Rājasuya*, [23] and *Rāmāyaṇa* speaks of gifts of jewels made by merchants[24]. The mention of jewels in several examples of our text[25], shows that the jewellers had a flourishing trade. *Suvarṇa* [26] (gold) is mentioned in most of the examples of our text. So, goldsmiths must have had a busy time satiating the demand of the people for gold ornaments.

In the Manuscript, there is also the mention of *Loha* [27]. *Loha* means iron, or something made of iron[28]. In our text, there are problems pertaining to the processing of iron[29] and about refining of iron[30]. It must be the crude form of iron being processed by blacksmiths. It was a common practice to use iron tools for killing animals and for agricultural purposes. The same tools and weapons must have been manufactured for the same purpose in the area represented by our Manuscript. The processing of iron presupposes the existence of blacksmiths. Thus, blacksmithery must have been one of the important occupations of the section of people at the time of our Manuscript.

Another metal of common utility that attracted the attention of our writer is

Mala [31]. Mala means a kind of brass or bell-metal[32]. Kaye has translated *Mala* as bronze. There is an example in our text that mentions the processing and the refining of *Maladagdha* or burnt bronze[33]. Thus processing and refining of the said metal, shows that the occupation of several people must have been connected with the making of Bronze utensils and implements.

Ambha-loha [34] is also mentioned in our text. Dr. Hoernle[35] takes the word *Ambha-loha* as Sanskrit *abhra-roha* and suggests its meaning as 'lapis-lazuli'. Lapis-lazuli is a semi-precious stone of a rich blue colour, consisting of lazurite and other minerals and used for jewellery, ornaments and pigmentation[36]. It was also famous for its medicinal value. Lapis-lazuli was produced in Badakshan, the only area in the world to produce it. It has also been used in the stone-work of Taj-Mahal. As per the Taj-Museum records these stones used to come from Afganistan. Since, these areas were conti. guous to Gandhāra, its mention in our text is naturally justified. Thus lapis-lazuli used as an ornament must have had engaged several people in its trade.

It won't be out of place to mention the animal-merchants, as our writer refers to them in an example[37]. It showed that some people were engaged in animal-business.

In our text, merchants not only appear in money-transactions but in one case, with *Brāhmaṇs* and others, as recipients of propitiatory gifts[38].

In the *Bakhshāli-Manuscript* [39], there is the mention of *'Arjjakas'*. *Arjjaka* means one who acquires or gets. Also, it means an acquirer or one who gains and acquires[40]. Kaye has translated it as 'banker'. Thus the mention of banker, shows that the trade at that time was flourishing so much that it resulted in such a rich trader class. Bankers must have pursued the profession of lending money and also, keeping the money of people in their safe-custody.

POSITION OF WOMAN

About the position of woman in the society, represented by our text, our information gleaned from the text is extremely scanty. We have an interesting reference in one of our examples, which shows that **Yuvī** (young women) also worked as labourers on daily wages. We quote the text of the example[41] below -

Sthāpanaṁ kṛiyate

....	1	*yuvī*	1	*śūdha*	1	*dṛishya*	20
	1		1		1		
....	3	*maṁ*	1	*maṁḍa*	1	*maṁḍe*	20
			1		2		
			2				

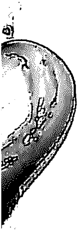
.... *ta datta jātam maṁda 2 yu 5 śūdhe* ....

FOODS AND DRINKS

In the *Bakhshāli Manuscript*, there are several names of food-stuffs and other commodities. The staple food of the people seems to have been *Śāli* [42]. *Śāli* means paddy or rice in general, but especially in two classes; one like white rice growing in deep water, and the other a red sort, requiring only a moist soil; there are also a great many varieties of this grain[43].

Godhuma [44] and *Yava* [45], these are also mentioned in our text. *Godhuma* means wheat[46]. *Yava* means barley[47], a chief food crop of the region from Vedic times. Thus besides more popular rice, barely and wheat were other chief food-grains.

As regards pulses there is mention of *mudga* [48] in our text. *Mudga* means Kidney-



bean[49], *moong* in vernacular and it seems to have been one of the ingredients of diet.

Lavaṇa is also mentioned in our text[50]. *Lavaṇa* means saline, or salt (especially sea-salt, rock or fossil-salt; but also factitious salt or salt obtained from saline earth)[51]. In our case it means rock-salt that was obtained from the Salt-Range in the region. Its mention shows that *Lavaṇa* must have been a commodity used in daily foods. The heaps of *Lavaṇa* are mentioned in our text.

In addition to these food-stuffs, honey derived from large bees called *Mākṣika* [52], and *Guḍa* [53] which means molasses have also been referred to by our author.

Apart from these food-stuffs, *kumkum* [54], one of the famous products of Kashmir, has also been mentioned. *Kuṁkuṁ* means Saffron (the plant and the pollen of the flowers)[55]. Saffron has been the monoply of Kashmir from ancient times[56]. It is grown on the vast plateau of Pampur (ancient Padmapura) and is used mainly as pigment, condiment and medicine; and enjoys an honourable position among the articles used by the Hindus in their daily worship[57]. It could be easily procured by the people of neighbouring Gandhāra.

DRINKS

An example of our text refers to the drinking of *madhu* [58]. *Madhu* means honey (said to possess intoxicating qualities and to be of eight kinds) according to the *Ṛigveda* [59]; In the *Grihya* and *Śrauta sūtra*, it means the juice or nectar of flowers, and sweet intoxicating drink, wine or spirituous liquor[60]. Madhu also means according to some explanations wine or spirit distilled from grapes[61]. Drinking of wine is recommended in the *Nila-mata purāṇa* [62] and also in the *kuṭṭanimata* of Damodargupta[63].

In our text there is mention of a traveller drinking *madhu* (wine) on the way to his

destination. It shows that wine was one of the famous drinks at that time and must have been lavishly consumed by the people. As the example quoted would show wine was an important item carried by travellers on their journey.

ANIMALS

In the *Bakhshāli Manuscript*, all the allusions to animals are of a general character. There is generally the mention of domestic animas. In a problem[64], there is the mention of *Aśva, Haya* and *Uṣtra. Aśva* means a horse, stallion. The horses are said to have seven breeds, so symbolical expression for the number 'seven' (that being the number of the horses of the sun)[65]. *Haya* means a horse in the *Ṛgveda*[66]. It is also a symbolic expression for the number 'seven' (on account of seven horses of Sun)[67]. The two terms *aśva* and *haya* are sometimes used as synonyms and the horse of an army has been called turga[68].

The other animals mentioned are *uṣtra*[69] or camels, *mahiṣi*[70] or buffalo, *gāvah*[71] or cows. While *turga* (horses) and *gaja*[72] (elephants) were used in the army, *uṣtra* (camels) seem to have been used for carrying loads. The cows and buffaloes were obviously the domestic animals reared for yielding milk. Buffaloes were also used for ploughing and driving carts. The birds are mentioned number of times under the term *khaga*[73]. In particular we find mention of only one bird *'griddha'*[74]. In a fragmentary problem[75] we have the expression '*tāni yata śara-paramparay ārjun-ena griddhra......*'. *Griddha* denotes a vulture, which from the days of *Rāmāyaṇa* has been a bird of reverence for the Hindus. However, they subsisted on the flesh of the dead animals as they do today. Owing to the fragmentary condition of the text, it is not clear in what connection they have been mentioed with Arjuna. However, the expression probably connotes the sense that (so many) vultures (were killed) by Arjuna with the series of arrows.........'.

Among the reptiles and insects, we find mention of *Nāga* [76], *Sarpa* [77] and *Kiṇa* [78]. *Sarpa* is mentioned in an interesting problem which states that 'A *sarpa*, eighteen *hastas* long enters its hole at the rate of one half plus one nineth of that minus one twenty first part of an *aṅgula* a day. In what time will it have completely entered its hole. It is evident that *sarpa* here denotes a snake. *Nāga* is mentioend in a problem which states that a Naga which is 100 *yojanas*, 6 *krośas*, 3 *hastas* and 5 *aṅgulas* long, sheds its skin at the rate one *aṅgula* in two days. In what time will it cast away its entire skin. The dimensions of the reptile given in the example would show that *Nāga* here represents a big snake or python. In the following fragmentary problem, we have the mention of *Kiṇa* (a worm), *kiṇa kilārdhāṁgulaṁ divase divase.......*

WEIGHTS AND MEASURES

The measures whether of time, length, capacity, weight or money employed in our text are almost the same as used in other Indian texts of the same nature. We discuss below these measures in detail and also compare them with the measures employed in other important arithmetical texts.

MEASURES OF TIME

The measures of time employed in our text are orthodox in nature. They are the usual measures of time in use in India from ancient times. The unit of time measure is '*ghaṭika*' [79], which as we know is equivalent to 24 minutes. *Ghaṭika* means a *muhurta* or thirtieth part of a day and night[80].

2 *Ghaṭikas*	=	1 *Muhurta*
30 *Muhurtas*	=	1 *Dina*
30 *Dinas*	=	1 *Masa*
12 *Masas*	=	1 *Varṣa*

Śrīdhara gives the following measures of time. The unit used by him is '*Ghaṭika*'.

60 *Ghaṭikas*	=	1 *Dina*
30 *Dinas*	=	1 *Masa*
or		
1800 *Ghatikas*		
12 *Masas*	=	1 *Varṣa*

Bhāskara uses the smaller measures of time, starting from *'Truṭis'*. *Truṭi* means a very minute space of time, equal to 1/4 of a *kṣaṇa* or 1/2 of a *Lava*, an atom (=7 *renus*)[81].

100 *Truṭis*	=	1 *Tatpara*
30 *Tatparas*	=	1 *Nimeṣa*
18 *Nimeṣas*	=	1 *Kaṣṭa*
30 *Kaṣṭas*	=	1 *Kalā*
30 *Kalās*	=	1 *Ghaṭika*
2 *Ghaṭikas*	=	1 *Kṣaṇa*
30 *Kṣaṇas*	=	1 *Dina*

In the *Viṣṇu-Purāṇa*, we have the following table of time. The smallest unit is *'Nimeṣa'*. *Nimeṣa* actually means 'twinkling of eye', hence twinkling of the eye considered as measure of time[82].

15 *Nimeṣas*	=	1 *Kaṣṭa*
30 *Kaṣṭas*	=	1 *Kalā*
30 *Kalās*	=	1 *Muhurta*
30 *Muhurtas*	=	1 *Dina*

Mahavira gives the following table. The smallest unit is *'Ucchavāsa'*. *Ucchavāsa* means breath, exhalation or breathing out[83].

7 *Ucchavāsas*	=	1 *Stoka*

7 *Stokas*	=	1 *Lava*
38 $\frac{1}{2}$ *Lavas*	=	1 *Ghaṭika*
2 *Ghaṭikas*	=	1 *Muhūrta*
30 *Muhūrtas*	=	1 *Dina*
15 *Dinas*	=	1 *Pakṣa*
2 *Pakṣas*	=	1 *Māsa*
2 *Māsas*	=	1 *Ṛitu*
3 *Ṛitus*	=	1 *Āyaṇa*
2 *Āyaṇas*	=	1 *Varṣa*

It is significant to note that our text contains no reference to hour and week although the former was used for astrological purposes and the latter came in general use in the early centuries of the christian era. Our author appears to have used the orthodox meausres.

MEASURES OF LENGTH

The following measures of length are used in our text. The smallest unit is *'Aṅgula'*[84]. *Aṅgula* means a fingers' breadth, twelve *angulas* making a *vitasti* or span[85].

24 *Aṅgulas*	=	1 *Hasta*
4000 *Hastas*	=	1 *Kroṣa*
8 *Kroṣas*	=	1 Yojana

The other measures of length used are *dhanu*[86], *gavyuti*[87] and *yava*. *Dhanu* means a measure of four hastas or cubits[88]. *Gavyuti* means a linear measure, a distance of about four miles[89]. *Yava* means the measure of a barley-corn, considered as equal to six mustard seeds[90]. Also, *Yava* is a measure of length equal to 1/6 to 1/8 of an aṅgula according to the Varāhmihira's *Bṛihat-saṁhitā*[91]. The ratios

given are as follows :

6 *Yavas*	=	1 *Aṅgula*
24 *Aṅgulas*	=	1 *Hasta*
4 *Hastas*	=	1 *Dhanu*
1000 *Dhanus*	=	1 *Krośa*
4 *Krośas*	=	1 *Gavyuti*
2 *Gavyutis*	=	1 *Yojana*

i..e 6 x 24 x 4 x 1000 x 4 x 2

or 4608000 *Yavas* = 1 *Yojana*

The measures of length in other texts are as follows. In the *Mārkandeya-Purāṇa,* the measures of length are given as under :-

8 *Yavas*	=	1 *Aṅgula*
6 *Aṅgulas*	=	1 *Pada*
2 *Padas*	=	1 *Vitasti*
2 *Vitastis*	=	1 *Hasta*
4 *Hastas*	=	1 *Daṇda*
2 *Daṇdas*	=	1 *Nādi*
2000 *Nādis*	=	1 *Gavyuti*
4 *Gavyutis*	=	1 *Yojana*

In the Mahāvīra's *Gaṇita-sāra-saṅgraha* the measures of length are given as below :-

8 *Sesamums*	=	1 *Yava*
8 *Yavas*	=	1 *Aṅgula*
6 *Aṅgulas*	=	1 *Pāda*
2 *Pādas*	=	1 *Vitasti*
2 *Vitastis*	=	1 *Hasta*

4 *Hastas*	=	1 *Daṇḍa*
2000 *Daṇḍa*	=	1 *Krośa*
4 *Krośas*	=	1 *Yojana*

i.e. 8 x 6 x 2 x 2 x 4 x 2000 x 4

or 6,144,000 *Yavas* = 1 *Yojana*

The Śrīdhara's *Gaṇitasāra* gives the following measures of length.

24 *Aṅgulas*	=	1 *Hasta*
4 *Hastas*	=	1 *Danda*
2000 *Dandas*	=	1 *Krośa*
4 *Krośas*	=	1 *Yojana*

Bhāskara in his *Līlāvati* gives the same as under :-

8 *Yavas*	=	1 *Aṅgula*
24 *Aṅgulas*	=	1 *Hasta*
4 *Hastas*	=	1 *Daṇḍa*
2000 *Daṇḍas*	=	1 *Krośa*
4 *Krośas*	=	1 *Yojana*

or 8 x 24 x 4 x 2000 x 4

or 6,144,000 *Yavas* = 1 *Yojana*

It will be noticed that both Mahāvīra and Bhāskara give 6,144,000 *Yavas* = 1 *Yojana* while our text gives - 4,608,000 *Yavas* = 1 *Yojana*. The difference in the change ratios is noteworthy.

THE MEASURES OF CAPACITY -

The measures of capacity used in our text are again the same as used in other texts. The unit is Pala[92]. Pala means straw, husk; a particular measure of fluids; a

particular measure of time[93]. Also according to Nirukta by Yāska it means a particular fluid measure[94].

2 *Palas*	=	1 *Prastha*
2 *Prasthas*	=	1 *Kudava*
4 *Kudavas*	=	1 *Prasrathi*
4 *Prasrathis*	=	1 *Aḍhaka*
4 *Aḍhakas*	=	1 *Drona*
16 *Dronas*	=	1 *Khāri*

The other texts give the following measures of capacity -

In *Atharva-Veda*, the smallest measure of capacity used is *'Krişṇala'*. *Krişṇala* is same as rati or *guñjā*. One *Krişṇala* is regarded by the *Krtyakalpataru* as equal to three *guñjas* or *raktikas* apparently through confusion[95].

5 *Krişṇalas*	=	1 *Māşaka*
64 *Māşakas*	=	1 *Pala*
32 *Palas*	=	1 *Prastha*
4 *Prasthas*	=	1 *Aḍhaka*
4 *Aḍhaka*	=	1 *Drona*

In *'Kautilya-Arth-Śāstra'*, the unit is *'Kudumba'*

4 *Kudumbas*	=	1 *Prastha*
4 *Prasthas*	=	1 *Aḍhaka*
4 *Aḍhakas*	=	1 *Drona*
16 *Dronas*	=	1 *Khāri*

In Varāha-Mihara's *Bṛihat-Saṁhita*, the unit is *pala*-

1. Dry - measure

4 *Palas*	=	1 *Kudava*
4 *Kudavas*	=	1 *Prastha*
4 *Prasthas*	=	1 *Āḍaka*

2. Liquid - measure

8 *Palas*	=	1 *Kuḍava*
4 *Kuḍavas*	=	1 *Prastha*
4 *Prasthas*	=	1 *Aḍhaka*

Mahavira has taken *Śodaśika*, as the lowest measure of capacity in his work *'Gaṇita-Sāra-Saṅgraha'*. *Shodaśika* is the name of a coin which may have been 1/16 of the standard coin in weight or value [96].

4 *Sodaśikas*	=	1 *Kuḍaha*
4 *Kuḍahas*	=	1 *Prastha*
4 *Prastha*	=	1 *Aḍhaka*
4 *Aḍhakas*	=	1 *Drona*
4 *Dronas*	=	1 *Māni*
4 *Mānis*	=	1 *Khāri*

CURRENCY

The most commonly used denominations of money used in our text are *dramma* [97] and *dināra* [98]. Their relationship with each other, however, is not given.

The term *Dināra* derived from the Roman *'denarius'* was originally a gold coin widely in circulation during the Greek and Kushan rule in India. The term was adopted by Guptas and was equal to 16 silver *rupakās*. In medieval ages, however, it signified a copper coin as is indicated by its profuse use in this sense in Kalhaṇa's *Rājtaraṅgiṇi*. In our text also, it probably denoted a copper coin for, a days wages

is stated to be from 1.5 to 3 *dināras* (folio 60).

The term *dramma* which is generally believed to be the derivative of Greek *drachma* was a coin denomination prevalent all over northern India "in the late medieval period, that is from 9th to the 13th century", according to D.R. Bhandarkar[99]. In Smith's opinion "the earliest record where this word has been traced is the Gwalior inscription of Bhojadeva of the Imperial Pratihara dynasty and dated 875 A.D.[100] However, the term occurs even earlier in a Yaudheya coin of 3rd century A.D., which bears the legend "*devasya dramma Brāhmaṇa*, which according to S.K. Chakarborty[101] may be construed as "*Brāhmaṇya devasya dramma*" meaning the coin dedicated to *Brāhmaṇyadeva* or *Kārtti-keya* the tutelary deity of the *Yaudheya* tribe[102]. The term also occurs in a couple of early medieval inscriptions from Himachal Pradesh. It is mentioned in the Luj (District Chamba, Himachal Pradesh) fountain incription of the first year of Jasata dated 1105-06 A.D. and in the *Baijnath* (District Kargra, H.P.) *Praśastis* No.2 dated A.D. 1204. In the former, owing to language being extremely corrupt, it is not exactly known as to in what connection it is mentioend. Vogel thinks that the *mula* (Sanskrit-*mūlya*) 20 dramma occuring in the record denotes the cost of grain (mentioned as *danika* or *dhanya*) supplied by donor for a feast held on the occasion of the erection of the fountain slab. In the *Baijnath Praśastis* it is mentioned in connection with the donations made to the Śiva temple. It is stated that the ruling chief Lakṣamaṇa Chandra allotted daily 6 *drammas* collected at the custom house called *maṇḍapikā*. It would sum that in our text as also in the records mentioned above *dramma* signified a coin in silver. Bhandarkar also takes the term to signify only the coins in silver[103]. Dr. Hoernle remarks "the way in which the two terms are used in *Bakhshāli Arithmetic* sems to indicate that the gold dināra and the silver *dramma* formed the ordinary currency of the day[104].

The two other coin-denominations mentioned in our Manuscript are *rūpa* [105] and *satera* [106]. *Rūpa,* also called *rūpaka,* is the same as dramma and denotes a silver coin. It was equivalent of 1/16th of the gold *dināra. Rūpa* is a particular coin (probably a rupee) according to Varāhmihira's *Bṛhat-Saṁhitā.* It is frequently mentioned in literature and inscriptions and denotes a silver coin. The silver coins of the Guptas and the Kalchuri King Kṛṣṇarāja were called the *rūpakas.* There is no reason to doubt that the term is used in our text also in the same sense.

Satera which is also sometimes spelt as Sateraka is probably the same as Greek 'Stater' and regarded as equal to two *dināras* [107]. Kaye, however, regards the significance of *Satera,* as it occurs in text, of doubtful import[108].

The term *Kākinī* [109] (cowrie) also occurs in our Manuscript Kākinī is name of a small coin, equal to 20 cowrie-shells according to Lilavati, and 1/4 of a paṇa according to *Krtyakalpataru* [110].

The mention of Kākini presuppores the existence of cowrie-shells as 20 cowrie-shells were equivalent to one Kākini. Cowrie-shells served as a medium of exchange from remote antiquity and even after the invention of metallic currency they continued to be used in day-to-day transactions side by side with coined money.

MONEY MEASURES

Our information regarding the money measures in use in the region represented by our Manuscript is extremely scanty. The coin denominations *dinara* and *dramma* are mentioned frequently as also occasionally *kākini* but their relationship with one another is no where given in our text. It is to be understood that in the early and medieval times measures of money were the same as the weight measures of different metals. The scraps of information as gleaned from our text are given below :-

i. *chhe°* 80 *rakti°* - *su°*, which means 80 raktikas = 1 *suvarṇa*.

ii. *chhe* 9 *gu°-va°*. It is not clear for what do the abbreviations *gu°*, and *va°* stand for. If we take *gu* to stand for *gunja* and *va* for *valla*, the expression would mean 9 *guñjas* = 1 *valla*.

iii.

1	1	1	1	1	1	1	108
1	2	2	2	2	2	2	1

pha dī *dhā* 8 *aṁ* 1.

which means $1 : \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$: : *idi°*+8 *dhā°*+1 aṁ°, where *di* stands for *dināra*, *dhā* for *dhānaka* and *aṁ* for *aṁsa*.

As per the statements given in the text as e.g. on folios 33 and 49 -

4 *aṁśas*	=	1 *dhānaka* and
12 *dhānakās*	=	1 *dīnāra*

We give below the table of money measures as found in our Manuscript and compare the same with those of Bhāskara and Śrīdhara.

In *Bakhshāli Manuscript*, the unit is *Raktikā*-

5 *Raktikās*	=	1 *dhānaka*
6 *Dhānakas*	=	1 *dramma*
2 *Drammas*	=	1 *dinara*
$1\frac{1}{3}$ *dināras*	=	1 *suvarṇa*

which gives $5 \times 6 \times 2 \times 1\frac{1}{3}$ or 80 *Raktikās* = 1 *suvarṇa*.

In Bhāskara's *Līlāvatī*, the unit is *varāṭaka* -

20 *Varāṭakas*	=	1 *kākinī*
4 *Kākinīs*	=	1 *paṇa*
16 *paṇas*	=	1 *dramma*
16 *drammas*	=	1 *niṣka*

In Śrīdhara's *Gaṇita-sāra* also, the unit is *Varāṭaka*

20 *Varātakas*	=	1 *Kākinī*
4 *Kākinīs*	=	1 *paṇas*
16 *Paṇas*	=	1 *puraṇa*

MEASURES OF WEIGHT

The measures are generally expressed by the abbreviations *mū, pā, ka, si, ya, ra* or *guin, aṁ, dhā* or *mā, to, pa, bhā* which respectively stand for *mudrika, pāda, kalā, siddhārtha, yava, raktikā* or *guñja, aṇḍika, dhānaka, tola, pala, bhāra.*

From various examples given in the *Bakhshāli Manuscript*, we get the following table of measures. *The unit is mudrikā.*

4 *mudrikās*	=	1 *pāda*
4 *Pādas*	=	1 *kalā*
$2\frac{1}{2}$ *Kalās*	=	1 *siddhārtha*
$2\frac{1}{2}$ *Siddhārthas*	=	1 *Yava*
$3\frac{1}{5}$ *Yavas*	=	1 *Raktikā*
$1\frac{1}{4}$ *Raktikās*	=	1 *Aṇḍikās*
4 *Aṇḍikās*	=	1 *dhānaka*
6 *dhānakas*	=	1 *dramma*

2 *drammas*	=	1 *tola*
8 *tolas*	=	1 *pala*
2000 *Palas*	=	1 *bhāra*

In Varāhmihira's *Bṛhatsaṁhitā* we have the following table, the unit is *mudri* -

4 *mudris*	=	1 *pāda*
4 *pādas*	=	1 *kalā*
$6\frac{1}{4}$ *kalās*	=	1 *yava*
4 *Yavas*	=	1 *andi*
4 *Andis*	=	1 *māṣa*
16 *māṣas*	=	1 *suvarṇa*

In Śrīdhara's, *Gaṇitsāra*, we have the following table, the unit is *Guñja*. *Guñja* means a unit of measurement, about 1.825 grains or .119 grammes in weight and is same as *Raktikā*-

5 *Guñjas*	=	1 *māṣa*
16 *Māṣas*	=	1 *karṣa*
4 *Karṣas*	=	1 *pala*

In Bhāskara's *Līlāvatī*, the unit is *yava*. *Yava* [111] means a barley-corn as a measure of weight = 6 or 12 mustard seeds = 1/2 *guñja*. *Yava* [112] also means a particular weight = 1/15 of *māṣa*; 1/3 of *rati*. Bhaskara gives the following table -

2 *Yavas*	=	1 *Raktikā*
3 *raktikas*	=	1 *valla*
8 *vallas*	=	1 *dharaṇa*
2 *dharaṇas*	=	1 *gadyāṇaka*

SOURCES OF REVENUE

Among the sources of revenues, we find mention of *śulka* [113] and the officer responsible for collection of *śulka* is called *śaulkika* [114]. *Śulka* as a fiscal term occurs in such early works as the *Atharvaveda* [115], *Dharamsūtras* [116] and the *Aṣṭādhyāyi* [117]. In the Amarkośa it is explained as *ghaṭṭādideya*, i.e. duties paid at the ferries, etc. Kṣirasvāmin commenting on the expression *ghaṭṭādideya* takes *śulka* to denote the ferry duties, the tolls paid at the military or police stations and the transit duties paid by the merchants[118]. The term also occurs in Manu[119] and is explained by the commentators as duties paid by the merchants. The *Arthaśāstra* [120] mentions the term quite frequently and from several references to it in the said text it may be explained as custom or toll duties levied on merchants and collected at the ferries, at the custom houses or octroi posts stationed at the main gate of the town, at the ports and at the frontier stations.

The *śulka* or toll-duties were collected at the toll-houses called *śulka-sālais*. We have definite information about levying of toll-duties on cloth, saffron and honey.

ii. POLITICAL THEORY AND ADMINISTRATION

The *Bakhshāli Manuscript* of which only a few leaves have been preserved completely, furnishes little information about the polity and government as it obtained in Gandhāra in the age represented by the text. We find mention in general of the term *'Nrip'* [122], which would denote the king.

We also find the mention of *'Rājaputra'* [123] which usually denotes a royal prince, but in the context in which it is mentioned in our text, it can only be taken to mean a Rājput claiming descent from the ancient *Kṣ atriyas*, employed in the service of the king. The context in the problem is as follows-

udaº || *rājputro dvayo kechi nṛipatis sevya santi vaiḥ mekāsyāhne dvayash shaḍ bhāga dvitiyasya divardhakam* |
prathamena dvitiyasya daśa dināra dattavān kena kālena samatāṁ gaṇayitvā vadāsū me ||

OFFICIALS -

We have reference in our text to an important army official named *Rākṣaka* [124] in the following example-

yady eka puruṣasya drammāṣ-śat... trimśabhir dinai jiva - lokā | *tat kāryaṁ prastutaṁ.... ssaptatināṁ..... pāka rākṣakānām drammaiṣ-ṣaḍbhi kati dinā jiva - lokaṁ bhavati...*

The *Rākṣakas* were probably the guardians of the fort. Forts were of great importance in ancient warfare and their protection was of supreme importance for the defence of the state.

There is a mention of another official called '*Śaulkika*' in the following example-

udaº || *mākṣikag-ghaṭakasyaiva dvi-tri-bhāga pravardhitaṁ dvitiye dvi-paṁchamo-bhāgo tritiye dvi-saptakodbhavaṁ chaturthe dvi-navaṁ-bhāgaṁ evaṁ jāta pala trayam* |
babhuva śaulkikai hṛitva kiṁ sarvaṁ vada paṇḍita ||

Śaulkika was a revenue official responsible for the collection of *śulka* (tax). The meaning and significance of this term *śulka* has already been discussed above.

ARMY -

Of the four traditional limbs of the army *Chaturaṅga bala*, our text[125] mentions chariots, elephants, footmen and horsemen in the ratios of 1:1:5:3.

The different divisions of army are as follows. There was probably more information given in the text for the terms *akṣauhinī, anikinī, chamū* and *pritāna*. According to dictionaries-

1 *akṣauhinī*	=	10 *anikinī* or 218, 700 in all.
1 *chamū*	=	129 chariots, 129 elephants, 2187 horses, 3645 footmen or 6090 (i.e. $3 \times 43 + 3 \times 43 + 3^7 + 3^6 \times 5 = 10 \times 3 \times 7 \times 29$).
1 *pritāna*	=	$243 + 243 + 729 + 1215 = 2430$ (or $3^4 + 3^5 + 3^6 + 5 + 3^5 = 10 + 3^5$)

Albiruni[126] gives the following scheme which is identical with that given in our text :

Each *akṣauhinī* has 10 *anikinī*

Each *anikinī* has 3 *chamū*

Each *chamū* has 3 *pritāna*

Each *pritāna* has 3 *vāhini*

Each *vāhini* has 3 *gaṇa*

Each *gaṇa* has 3 *gulma*

Each *gulma* has 3 *senamukha*

Each senamukha has 3 *patti*

Each *patti* has 1 *ratha,*

and a *ratha* comprehends besides, one elephant, three riders and five footmen.

REFERENCES

1. *The Bakh Ms.* folio 9 recto, p. 173.
2. According to Apte. V.S. *The Practical Sanskrit-English Dictionary*, Reprint, Delhi, 1985, p. 705.
3. Majumdar R.C. The History and Culture of Indian People, *The Age of Imperial Unity*, Bharatiya Vidya Bhawan, Vol. II, p. 542.
4. *The Bakh. Ms.* folio 60 verso, p. 217.
5. According to Wilson, H.H. *Sanskrit-English Dictionary*, Delhi, 1979, p. 416.
6. *The Bakh. Ms.* folios 10 recto and 10 verso, p. 198.
7. *The Bakh Ms.* folios 44 recto, p. 234.
8. *The Bakh Ms.* folios 13 recto, p. 203 and 37 recto, p. 227.
9. According to Wilson, *Sanskrit-English Dictionary*, p. 228.
10. *The Bakh Ms.* folio 50 recto, p. 237.
11. *The Bakh Ms.* folio 9 recto, p. 173.
12. Accordingly to Apte, The Practical Sanskrit-English Dictionary, p. 863.
13. Ray, S.C. Early History and Culture of Kashmir, New Delhi, Second Revised Ed. 1970, pp. 100-101.
14. *The Bakh. Ms.* Folio 61 recto, p. 218.
15. According to Williams, M. Monier, *A Sanskrit-English Dictionary*, New Ed., Enlarged and Improved by Leumann and Cappeller, Delhi, 1986, p. 325.
16. *The Bakh. Ms.* folio 9 verso, p. 174.
17. *The Bakh. Ms.* folio 14 verso, p. 205.
18. *The Bakh. Ms.* folio 15 verso, p. 207.
19. *The Bakh. Ms.* folio 3 verso, p. 171.

20. According to Monier Williams in *'A Sanskrit-English' Dictionary*, p. 915.

21. *The Bakh. Ms.* folio 1 recto, p. 168.

22. According to Wilson in *'Sanskrit-English' Dictionary*, p. 542.

23. *Mahābhārata;* Ed. Bhandarkar Research Institute Poona, II. 45.32.

24. *Rāmāyaṇa*, Ayodhya Kānda, 82.8.

25. *The Bakh. Ms.* folio 34 verso, p 226.

26. *The Bakh. Ms.* folio 11 verso, p. 200.

27. *The Bakh. Ms.* folio 10 verso, p. 199.

28. According to Apte in 'The practical Sanskrit-English Dictionary, p. 822.

29. *The Bakh Ms.* folio 14 verso, p. 206.

30. *The Bakh. Ms.* folio 10 verso, p. 199.

31. *The Bakh Ms.* folio 10 recto, p. 198.

32. According to Lexicographers (i.e. a word or meaning which although given in native lexicons, has not yet been met with any published text) in Monier Williams *"A Sanskrit-English Dictionary"*, p. 792.

33. *The Bakh. Ms.* folio 10 recto, p. 198.

34. *The Bakh. Ms.* folio 14 verso, p. 206.

35. Hoernle, R. The Bakhshāli Manuscript, *IA*. Vol. XVII, p. 48.

36. *The New Encyclopaedia Britannica*, Vol. 7, 15th Edition, 1985, p. 157.

37. *The Bakh. Ms.* folio 3 verso, p. 170.

38. *The Bakh. Ms.* folio 3 verso, p. 171.

39. *The Bakh. Ms.* folio 13 verso, p. 204.

40. According to Wilson in *'Sanskrit-English Dictionary'* p. 57.

41. *The Bakh. Ms.* folio 58 verso, p. 191.

42. *The Bakh. Ms.* folio 3 recto, p. 171.
43. According to Wilson in *'Sanskrit-English Dictionary'*, p. 715.
44. *The Bakh. Ms.* folio 3 recto, p. 171.
45. *The Bakh. Ms.* folio 3 recto, p. 171.
46. According to Monier Williams in *'ASanskrit-English Dictionary'*, p. 365.
47. According to Monier Williams in *'ASanskrit-English Dictionary'*, p. 847.
48. *The Bakh. Ms.* folio 30 verso, p. 213.
49. According to Apte in *'The Practical Sanskrit-English Dictionary,'* p. 765.
50. *The Bakh. Ms.* folio 36 recto, p. 225.
51. According to Monier Williams in *'ASanskrit-English Dictionary'*, p. 898.
52. *The Bakh. Ms.* folio 16 recto, p. 208.
53. *The Bakh. Ms.* folio 13 recto, p. 203.
54. *The Bakh. Ms.* folio 12 recto, p. 202.
55. According to Monier Wiliams in *'ASanskrit-English Dictionary'*, p. 287.
56. Bamzai, P.N.K. *Socio-Economic History of Kashmir*, New Delhi, 1987, p. 15.
57. *Ibid.*
58. *The Bakh. Ms.* folio 12 recto, p. 201.
59. According to Monier Williams in *'ASanskrit-English Dictionary'*, p. 779.
60. *Ibid.*
61. According to Wilson in *'Sanskrit-English Dictionary'*, p. 546.
62. *Nilmata-Purana*, Ed. by Ramlal Kanjilal and Jagdhar Zadoo, Lahore, 1924; verses, 465, 523, 676. Republished by Ved Ghai, Srinagar, 1968.
63. Shastri, A.M., *India as seen in The Kuṭṭaṇī-Mata of Damodargupta*, Delhi, 1975, p. 132.

64. *The Bakh. Ms.* folio 3 verso, p. 170.

65. According to Monier Williams, *op. cit.* p. 114.

66. *Ibid.;* p. 288.

67. *Ibid.*

68. *The Bakh. Ms.* folio 47 recto, pp. 228-229.

69. *The Bakh. Ms.* folio 3 verso, p. 170.

70. *The Bakh Ms.* folio 35 verso, p. 163.

71. *Ibid.*

72. *The Bakh. Ms.* folios 47 verso and recto, pp 228-229.

73. *The Bakh. Ms.* folio 34 recto, p. 226.

74. *The Bakh. Ms.* folio 34 verso, p. 226.

75. *Ibid.*

76. *The Bakh. Ms.* folio 32 recto, p. 224.

77. *The Bakh. Ms.* folio 20 verso, p. 222.

78. *The Bakh Ms.* folio 20 verso, p. 222.

79. *The Bakh. Ms.* folio 13 recto, p. 190.

80. According to Wilson in *'Sanskrit-English Dictionary'*, p. 253.

81. According to Apte in *'The Practical Sanskrit-English Dictionary'*, p. 486.

82. According to Wilson in *'Sanskrit-English Dictionary'*, p. 392.

83. According to Apte in *'The Practical Sanskrit-English Dictionary'*, p. 256.

84. *The Bakh. Ms.* folio 28 verso, p. 222.

85. According to Monier Williams in *'A Sanskrit-English Dictionary'*, p. 8.

86. *The Bakh. Ms.* folio 32 recto, p. 224.

87. The Bakh. *Ms.* folio 12 recto, p. 201.

88. According to Lexicographers (i.e. a word or meaning which although given in native lexicons, has not yet been met with any published text) in Monier Williams *'A Sanskrit-English Dictionary'*, p. 509.

89. *Epigraphia Indica*, Volume, XXII.

90. According to Wilson in *'Sanskrit-English Dictionary'*, p. 585.

91. According to Monier Williams in *A Sanskrit English Dictionary*, p. 847.

92. *The Bakh. Ms.* folio 16 recto, p. 208.

93. According to Apte in *'The Practical Sanskrit-English Dictionary'*, p. 605.

94. *Yāska's Nirūkta*, Ed. by Mukund Jha Bakshi, Bombay, 1930, XIV, p. 7.

95. *Krtyakalpataru*, vyavahāra-kānda, Ed. by K.V. Rangaswami Aiyangar, p. 125.

96. *South Indian Inscriptions*, Vol. IV, Ed. by Rao Bahadur H. Krishna Shastri, Madras, 1923.

97. *The Bakh. Ms.* folio 65 recto, p. 213.

98. *The Bakh. Ms.* folio 61 recto, p. 218.

99. Bhandarkar, *Lectures on Ancient Indian Numismatics*, Patna, 1921, reprinted, Delhi 1984, p. 206.

100. Smith, *Coins of Ancient India*, Catalogue of Coins in the Indian Museum, Varanasi, 1972, p. 186.

101. Chakraborty, S.K. Forein Denominations of Ancient Indian Coins, IHQ, Vol. XV, pp. 70-71, reprint, Delhi, 1985.

102. *Ibid*.

103. Bhandarkar, Lectures on Ancient Indian Numismatics, p. 207.

104. Dr. Hoernle, *IA*, Vol XVII, p. 37.

105. *The Bakh. Ms.* folio 21 recto p. 191.

106. *The Bakh. Ms.* folio 34 verso, p. 226.

107. *JNSI*, Varanasi, Vol. XV, p. 153.

108. *The Bakhshāli Manuscript*, Kaye, p. 64.

109. *The Bakh. Ms.* folio 36 recto, p. 225,

110. *Kṛtyakalpataru, op. cit.* p. 125.

111. According to Monier Williams, *op. cit.,* p. 847.

112. Sircar, D.C. *Indian Epigraphical Glossary*, Delhi, 1966, p. 442.

113. *The Bakh. Ms.* folio 14 verso, p. 205.

114. *The Bakh. Ms.* folio 16 recto, p. 208.

115. *Gautama Dharma-Sūtra*, X. 25; *Āpsatamba Dharma-Sūtra*, II. 26.9.

116. *Dharmasutra*, V, I. 47.

117. *Āṣṭādhyāyi*, II, 8.27.

118. *"Ghaṭṭo naditarasthānamādi-śabdat-gulmapratolyādau praveśya-naiśkramya-dravyabho rājagrāhya bhāgah śulkah"*

119. VIII. 307.

120. II. 6, 16, 21, 28, 35, etc.

121. *Epigraphia Indica*, Vol. I. p. 117.

122. *The Bakh Ms.* folio 70 verso, p. 185.

123. *The Bakh. Ms.* folio 61 recto, p. 218.

124. *The Bakh. Ms.* folio 65 recto, p. 213.

125. *The Bakh. Ms.* folio 47 verso and recto, p. 228.

126. *Albiruni's India*, English translation by E.C. Sachau, London, 2nd ed., 1910, Vol. I, Ch. XIVIII, reprinted New Delhi, 1983, pp. 407-408.

CHAPTER VII

i. RELIGIOUS CONTENT
ii. THE ASTRONOMICAL DATA

VII
THE RELIGIOUS CONTENT

The *Bakhshāli Manuscript* furnishes very meagre information of the contemporary religious conditions of the area of its provinance. However the references to religious matters, though very few in number are interesting as they throw welcome light on the religious beliefs of the contemporary people. We now proceed to discuss the religion as reflected in the arithmatical problems of our text.

ŚAIVISM

The provinance of the *Bakhshāli Manuscript* falls in the Gandhāra region which was the north-western most kingdom of the Indian sub-continent. The cult of Śiva in Gandhāra dates back to the remote anti quity when its popularity among the non-Aryans in pre-Vedic period is attested to by the archaeological finds in the Sindhu valley consisting of the prototypes of Siva as Pasupati and his emblem the Śiva-linga. The *Śatapatha-Brāhmaṇa* refers to the popularity of the worship of Bhava, a name of Śiva, among the Bāhlikas who lived in the present region of Balkh in Afganisan and the adjoining Gandhāra[1]. In the *Aṣṭādhyāyi* of pāṇinī (who hailed from Gandhāra) occurs the *Sūtra Sivādibhyo'n* (IV. 1.112) meaning that an affix aṅ comes in a sense of a descendant after the name Śiva etc. The early Greek writers like Strabo refer to the tribes of the Panjab and Gandhāra like Siboi and oxgdrakai as regarding themselves the descendants of Siva[2]. Magasthenese, the Greek ambassador in the court of Chandragupta Maurya (about 300 B.C.) refers to the popularity of worship of God Dionysios identified with Śiva in the hill regions of northern India which may have included Gandhāra as well.

The evidence of popularity of the cult of Śiva in Gandhāra in the centuries preceding the christian era is furnished by the early Indian coins hailing from Taxila which contain the theriomorphic and anthropomorphic figures of Śiva[3]. Some coins of the Indo-Greek king Demetrius, who ruled in Gandhāra about 200 B.C. bear the figure of Śiva's emblem, the trident on the reverse[4].

Despite the predominance of Budhhism in the region in the centuries following the Christian era and the patronage extended to it by the Kuṣāna rulers Śaivism continued to be a popular faith as is indicated by the coins of the Kuṣāna rulers Kadaphsis II, Kaniṣka, Huviṣka and Vāsudeva which contain the figures of Śiva and of his emblems like the trident and the sacred bull.

In the post-Kuṣāna period Śaivism appears to have flourished in Gandhāra under the patronage of the Sassanian rulers. A Kuṣāno-Sassanian gold coin issued under the sovereignty of Shahpur I (A.D. 256-264) bears the figure of Śiva in the Sassanian dress standing before the bull Nandi[5]. To the same period belongs an image of Śiva discovered from Charsada in Peshawar district depicting Śiva with three eyes, three heads and six arms standing before the bull Nandi[6].

The two principal sects of Śaivism, the Pāśupata and the Kāpālika are mentioned by Vasubandhu (who hailed from Peshawar) in the commentary of the *'Abhidharmakośabhāṣya'* a famous buddhist text of A.D. 4th century[7]. Another Buddhist text of the same period *'Mahāmāyūrī'* mentions Śiva as the presiding deity of Śivapura identified with *Udicyāgrāma* of the *Mahābhāṣya*[8] and the Siboi country in Gandhara mentioned by the classical writers.

In our text the name of Śiva[9] occurs on a scrap that appears to be the colophon of the text. Here Śiva is credited with the granting of gift of the science of calculation to the human race after the creation of the world. An important epithet of Śiva namely sulin[10] or the trident wielder occurs on folio 33 recto, where some individual

is described as making a gift of one *kalā* plus one *pāda* and one *yavana*, daily at the shrine of Sulin. Again, the name occurs in abbriviated form *Sū* in folio, 44 verso, where offerings to him are mentioned[11]. Śiva's consort has been mentioned in two different problems as Bhavāni[12] and Devi[13], where we have references to the gifts presented to her, too.

It would thus appear that Śaivism was a flourishing cult in ancient Gandhāra during the period represented by our record and offerings were daily made to Śiva and to his consort.

VAIṢṆAVISM

The worship of viṣṇu was popular in Gandhāra as early as the 5th century B.C. in the time of the great grammarian Pāṇinī. Pāṇinī in his *Aṣṭādhyāyī*[14]. gives a rule for the fomation of the word 'Vasudevaka' in the sense of "a person whose object of *bhakti* (devotion) is Vasudeva." The term vasudeva is interpreted by Pātanjali to stand for 'Vasudeva-kṛṣṇa or Viṣṇu'[15].

The prevalance of *Vaiṣṇava* faith in the 2nd century B.C. is testified by the evidence of an inscription of a Greek ambassador Heliodorus in which he describes himself as a resident of Taxila and a Bhāgvata or worshipper of Viṣṇu[16].

Vaiṣṇavism enjoyed popularity in the A.D. 2nd century under the patronage of Kuśāna king Huviṣka. Huviṣka, despite his leanings towards Buddhism was well disposed towards the Vaiṣṇava faith. Some of his coins bear figures of Viṣṇu and in a seal matrix attributed to him, he is represented as kneeling reverentially before Viṣṇu with his hands in anjali pose.[17]

Vaiṣṇavism must have enjoyed considerable popularity during the reign of Śāhi king Bhimdeva (10th century) who assumed the title of Gadāhasta (weilder of mace, a popular appellation of viṣṇu) as is indicated by an epigraphic record from Dewai (Gandhāra) belonging to him.[18] Owing to the paucity of material it is not

possible to trace the history of Vaiṣṇavism in Gandhāra in the subsequent periods.

However, an inscription of the Laukika year 538 corresponding to A.D. 1461, which is preserved in the Peshawar museum and which records the construction of a tank by a certain individual Vaṇhadaka, begins with an invocation of Viṣṇu.[19] The find spot of the inscription is not definitely known but it is said to hail from Hazara district.[20] If this be true, this record which contains an eulogy of Visnu, would furnish evidence of the popularity of the Visnu worship in certain areas in Gandhāra even as late as the 15th century.

Vaiṣṇavism also was a populoar cult in ancient Gandhāra during the period of our text as indicated by references to *Vāsudeva* and *Śeṣ a* mentioned as Mahoraga, the great serpent. Vasudeva is mentioned in folio 44, where some offerings to him by an individual are described.[21] Mahoraga is mentioned in folio 37, where the chariot of Sun is described as "surrounded by a group of Gods, Mahoraga, *Siddhas*, *Vidyādhars* (divine musicians)[22]. The connection between the sun and the sleep of God Viṣṇu on Śeṣ a is well known.

Surprisingly, we find no reference to Buddhism in our text, despite the fact that Gandhāra remained an important centre of Buddhism for centuries preceding and following the Christian era.

The popularity of the great Indian epics *Rāmāyaṇa* and *Mahābhārata* among the people of Gandhāra during the period represented by our text is indicated by number of references to the important characters of both the epics. Among the characters of the *Rāmāyaṇa*, we find references to Sītā, Rāvaṇa & Śatrudama. An interesting problem[23] occurs on folio 32, which is as under—

When Sītā had been carried up thirty yojanas into the air by Rāvaṇa, she dropped some thing to earth, which turned over eight times in $1\frac{1}{10}$ dhanūs, how many revolutions did it make before reaching earth. The reference here appears

evidently to be to the tying up of some jewellery in a garment and dropping the same to earth by Sitā in order to provide a due to the passage through which she was carried off through the air by Rāvaṇa.

The name Śatrudhama[24] occurs in the following fragment–|| *kaschid rāja kumāra Śatrudhama* | It means 'A certain prince Śatrudhama.' The phrase may as well mean 'a certain prince (engaged in) curbing (his) enemies'. (employed or fought so many soldiers). *Śatrudhama* also appears to be another name of Śatrughana, the younger brother of Rāma and Commander-in-chief of his army, both terms having the same meaning.

The acquintance with the *Mahābhārata* is understandable as the scene of its activity was not very far from Gandhāra and princess of Gandhāra and her brother Śakuni played important role in the epic. The characters of the Mahābhārata mentioned in our text are Ar juna also mentioned as Pārtha, Yuddhistra & Pāndu. These names are mentioned in connection with some arhmetical problems. Arjuna[25] is mentioned in a fragmentary problem as follows–

| tāni yata śara — Paramparay, ārjunena griddhra............' |

Pārtha[26] is mentioned in problem like the following–

"Pārtha pierced each soldier with $16\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)$ arrows and slew four divisions of the army. How many arrows did he use? A similar problem occurs twice in Bhāskara's Līlāvatī–

"Pārtha, irritated in a fight, shot a quiver of arrows to slay Karṇa. With half of his arrows he parried those of his antagonist; with four times the square-root of the quiver full, he killed his horses; with six arrows he slew Śalya with three he demolished the umbrella, and with one he cut off the head of the foe. How many were the arrows which Arjuna let fly?"

In an isolated fragment Yudhisthira[27] is described as a king belonging to the Pandu family

"rāja Yudhisthira nāma Pāndu vaṁśa.........."

Demi-Gods—

From pretty early times Indians believed in the existence of semi-divine spirits or *Devayonis*. Of them *Vidyādharas* and *Siddhas* are mentioned in our text. In the *Amarkośa*, they are classed as devayonis. In our text, they are described as surrounding the chariot of the sun. The problem[28] in which they are mentioned is as follows—

"The chariot of sun (Bhānu) is surrounded by the group of gods, great snakes, *Siddhas* and *Vidyādharas*. In a day and night its journey is said to be half a hundred kotis. Tell me, O best of calculators, how much in one muhūrata."

In the *Kuṭṭanimata* [29] also Śiva is said to have been attended by the demigods known as Vidyādharas.

The mythical mountain *Sumeru* is mentioned at one place in our text and described as the dwelling place of the gods. The problem[30], where the *Sumeru* is mentioned is as follows— "From the home of the gods a certain person desires to ascend swiftly Sumeru, the pole of the Earth and the dwelling place of gods. He goes constantly at the rate of seven times one and a half and its quarter with one-third and one-fifth. The height of Sumeru is eightly four thousand yojanas. In what time will he reach the summit? Give me well consideared answer." Ancient texts place it in the northern division of India.[31] B.C. Law takes it to be identical with the Rudra Himālaya in Garhwal where the Ganges rises.[32]

Religious Practices—

An analysis of our text throws some light on the religious tendencies of the age represented by it. In India the mundane world or *Sansār* has always been regarded as a place of misery and suffering and the next world or *Parloka*, as an abode of eternal joy and bliss. Gifts were made to *Brāhmaṇas* and to the images of gods and godesses, to secure relief from the miseries of the *Sansār* and bliss in *Parloka*. The following example[33] is noteworthy in this context—

"Daily earning $\dfrac{1\frac{1}{2}}{1\frac{1}{3}}$; given for *Bhavāni* 8 in $5\frac{1}{3}$ days; given for *Parloka* 1 in 32; given for Sulin $\dfrac{2\frac{1}{2}}{4 \times 36}, \dfrac{1}{260}$ years; reserve 700."

Food to the poor was given with the same objective in mind. This practice is alluded to in the following problem[34]. "The earning of dināras is difficult but consuming them is easy. One gives one-half increased by ration of one half (six times) for food for poor. What is the amount consumed in 108 days ?"

Astronomical Data

The astronomical data, found in our text is almost negligible. We find the mention of *Sūrya* and its synonym *Divākara* in the following fragmentary problem[35].

.... *Sūrya māṇasya*
divākarasya ghaṭikai kiṁ prayātasya vada
.... *niśchitam* ....

The problem may be roughly restored as 'The sun *(Sūrya)* traverses 500,000,000 *yojanas* in a day. State with certainly the amount of the journey of the Sun *(Divākara)* in a *ghatikā*. There is the mention of another synonym of sun i.e. *Bhānu* in another problem[36].

'Bhāno ratham sura mahoraga siddhasam (g) hai vidyādharai parivritaṁ ....
ahorātru | *koṭi śatardhaṁ sa rathaṁ pryāsyāt tad brūhi śastra kuśalo* ....
vaktuṁ || *muhūrtam ekena kiṁ gacche brūhi me ganakottamā* ||

The following fragmentary problem[37]. is probably related to Jupiter —

"... *bhāge bhaved rāśi* |

ūrdha chhedam 10800 viliptānam .... *liptā 5"*

While handling this problem, Kaye says – the remanant of a problem possibly related to the daily motion of Jupiter, which according to *Sūrya Siddhānta*, amounted to very nearly 5 minutes of arc (liptā). *Bhānuja* [38] (Saturn) is mentioned in the following problem.

If *Bhānuja* (Saturn) moves through a sign in two and a half years, state, O knower of the truth, what will its motion in a solar day *(vāsareṇa)* be equal to.

In this above mentioned problem, we find the mention of *Vāsara*, which means a solar day.

Pṛithivī (The Earth) has been mentioned in the following arithmetical problem[39], where *sumeru* (the dwelling place of Gods) has been described as the pole of the Earth—

uda° || *Sumeru pṛithivi Śaṁku surānāṁ parimāśrayam* ||
āga ām .... *kaśchi tarasā suramadiraṁ* ||
Satatam Sapta – Sārdhāṇām sa pāmadhya ....||
Satṛi-bhāgā tṛi paṁcaṁśa nityam evaṁ ca gacchati |
yojanānāṁ sahasrāṇicatur-āśitir ucchritam |
Kena kālena sau gacche vada me suniśchitaṁ ||

which means— "From the home of the gods a certain person desires to ascend swiftly *Sumeru*, the pole of the Earth and the dwelling place of the gods. He

goes constantly at the rate of seven times one and a half and its quarter with one-third and one-fifth. The height of sumeru is eighty-four-thousand yojanas. In what time will he reach the Summit ? Give me well considered answer.

REFERENCES

1. Chattopadhyaya, Sudhakara, *Evolution of Hindu Sects*, New Delhi, 1970, pp. 91, 95.
2. *Ibid*, pp. 94-95.
3. Banerjee , J.N. *Development of Hindu Iconography*, Calcutta, 1956, pp. 112 ff.
4. cf. Smith, V.A. *Coins of Ancient India*, Catalogue of the coins in the Indian Museum, p. 9.
5. Herzfeld, E. Kuśāno–Sassanian Coins, *Memoirs of the Archaeological Survey of India*, No. 38, Coin No. 7.
6. *Archaeological Survey of India*, Annual-Report, 1913-14, pp LXXII; Jaya Gosvami, Cultural History of Ancient India. p. 51.
7. Winternitz. M, *A History of Indian Literature*, Vol. II, Delhi, Revised ed. 1983, p. 355.
8. *Mahāmayuri*, V. 28, *Mahābhāṣya*, IV. 2.3., Levi, S. *Journal Asiatique*, 1915, pp. 21, 70.
9. *The Bakh. Ms.* folio 50 recto, p. 237.
10. *The Bakh. Ms.* folio 34 recto, p. 226.
11. *The Bakh Ms.* folio 44 verso, p. 233.
12. *The Bakh Ms.* folio 44 verso, p. 233.
13. *The Bakh. Ms.* folio 49 recto, p. 231.
14. *Aṣṭādhyāyi*, IV. 3.98.

15. Aggrawal V.S. *India as known to Pāṇinī,* Varanasi, Second ed. 1969, p. 359.

16. Besnagar Pillar Inscription, *Epigraphia Indica,* Vol. IX, Appendix. p. 63.

17. Majumdar, R.C. *The Age of Imperial Unity,* Vol. II, p. 439.

18. *Epigraphia Indica,* Vol. XXI, pp. 298 F.

19. *Epigraphia Indica,* Vol. X, pp. 80-81.

20. *Ibid* . p. 79.

21. *The Bakh Ms.* folio 44 recto, p. 234.

22. *The Bakh Ms.* folio 37 recto, p. 227.

23. *The Bakh Ms.* folio 32 recto, p. 224.

24. *The Bakh Ms.* folio 47 recto, p. 229.

25. *The Bakh Ms.* folio 34 verso, p. 226

26. *The Bakh Ms.* folio 47 verso, p. 228.

27. *The Bakh Ms.* folio 37 verso, p. 227.

28. *The Bakh Ms.* folio 37 recto, p. 227.

29. Shastri, Ajay Mitra, *India as seen in the Kuṭṭanimata* of Damodargupta, Delhi, 1975, p. 62

30. *The Bakh. Ms.* folio 33 verso, p. 223.

31. *Bṛhat–Saṁhita,* XIV. 24. Ed. with Battotpala's commentary by sudhakara Dvivedi, Varanasi, 1895-97.

32. Law, B.C. Historical Geography of Ancient India, Paris, 1954, p. 111.

33. *The Bakh. Ms.* folio 44 verso, p. 233.

34. *The Bakh. Ms.* folio 33 recto, p. 223.

35. *The Bakh. Ms.* folio 37 recto, p. 227.

36. *Ibid*

37. *The Bakh. Ms.* folio 37 recto, p. 227.

38. *Ibid.*

39. *The Bakh. Ms.* folio 33 verso, p. 223.

CHAPTER VIII

CONCLUSION

VIII

SCOPE AND IMPORTANCE OF THE BAKHSHALI MANUSCRIPT

The subject matter of the *Bakhshāli Manuscript* has been analysed in detail in the previous chapters. In this chapter, we shall discuss the scope and importance of the text as also its relationship with other important treatises on Mathematics.

The method of exposition discussed in detail above comprising the rule, the example, statement, solution and proof or verification cosiderably differs from what we find in other Indian mathematical treatises. In Brahmgupta, we find the rules followed by very few examples. No solutions have, however been given. Same is the case with Mahāvīra, who however, gives more examples illustrating a rule than are found in the Brahmagupta's work. The examples in Mahāvīra have been given the name *uddesaka* as compared to *udaharana* found in the *Bakhshāli Manuscript*. Śrīdhara (7th century A.D.) and Bhāskara (11th century A.D.) give the rule, the examples as also the statement. They, however have not recorded the solution of the examples but have given only the answers of the examples. The proofs or the verifications are the distinguishing feature of our text as no writer on Indian Mathematics is known to have given verification of the solutions of their examples.

The negative sign denoted by a cross and placed after the number affected is another distinguishing feature of our text as the same is not found employed elsewhere in any treatise on mathematics. Its origin and implication have been discussed in detail above.

Among the early Mathematical treatises, the method of finding the least common multiple as a part of the solutions of the problems containing fractions is

found only in our text. The method of finding the least common multiple is not found in the works of Āryabhaṭṭa, Brahmagupta and Bhāskara. It is however found in the *Gaṇita-sāra-sangraha* of Mahāvīra (850A.D.).

The arithmetical notation generally employed throughout the *Bakshāli Manuscript* is the decimal place-value notation. The exclusive application of this notation in our text is very much noteworthy in as much as in almost all the available Indian mathematical treatises, the word numerals have been copiously used.

The name *rūpoṇa karaṇa* denoting the rule for the summation of a series in arithmetical progression and discussed above in detail, is again unique for our text. It is not met with elsewhere. It is, however, to be noted that the technical terms which have been commonly used in our work in connection with arithmetical progression such as *ādi, prabhava, caya, uttara pada, dhana,* etc. are all the same as in other treatises.

The symbol to denote an unknown quantity in our text is '**o**', the details of which have already been discussed above. The same sign has been used to denote the unknown element in the statement of problems in the arithmetics of Śrīdhara and Bhāskara. Thus we have *ādhi*20/*u*o/gacchah 7/gaṇitaṁ 245/, which is a statement of an arithmetical progression where first term is 20, number of terms is 7, sum is 245 and where common difference is not known.

As stated above,in the Bakhshāli text two sides of an equation are written down one after the other in the same line without any sign of equality being interposed. For example— $\frac{x}{2} + \frac{x}{3} + \frac{x}{5} = 65$.

is represented as $\begin{array}{|c|c|c|c|c|} 1 & 1 & 1 & \text{dṛśya} & 65 \\ 2 & 3 & 4 & & 1 \\ \hline \end{array}$ in our text.

Śrīdhara and Bhāskara follow the same plan with the omission of lines by Bhāskara. In all the three treatises *dṛśya* sometimes abbreviated as *dṛ,* denotes the

absolute term. In the Bakhshāli text, the term refers to the "gives" while in other works it generally refers to the "remains". According to B.B. Datta, the term is closely related to *rūpa* meaning appearance" which is the name for the absolute term in the Indian Algebra.

Out text employes in general the same technial terms as are employed in other treatises but some marked differences in the employment of technical terms noticed in the text are noteworthy. Thus the term used to denote reductions of fractions to a common denominator in all treatises is *savarṇana* meaning "making of the same class". But in our text the term used for the same purpose is *sadṛśīkaraṇa* or making similar or *kara-sāmyakaraṇa* or making the denominations equal.

The other noticeable differences are the use of *sthāpanā* for statement in place of *nyāsa* used by other treatises, *varga* to denote series in lieu of *średhi* and *ruponakaraṇa* for summation of series which is exclusively used in our text and is not found elsewhere.

The four fundamental arithmetical operations, viz., addition, subtraction, multiplication and division are indicated as noted above by *yu, kṣa, gu* and *bhā* respectively, which are the abbreviations for Sanskrit words *yuta, kṣaya, guṇita* and *bhāga*, meaning addition, subtraction, multiplication and division respectively. This principle of choosing abbreviations of words to indicate fundamental arithmetical operations is unique to our text and is not found in any other arithmetical treatise.

Again in the *Bakhshāli Manuscript*, the square-root of a quantity is indicated by the accompanying symbol *mū*, which is an abbreviation for *mūla* meaning 'root', while in the rest of the Indian Mathematics it is indicated by *ka*, an abbreviation for *karaṇi*, meaning surd.

Hoernle points out to 'peculiar connection' between the *Bakhshāli text* and the

Brāhma-sphuṭa-siddhānta of Brahmagupta. He says, "There is curious resemblence between the fiftieth Sūtra of the Bakhshāli arithmetic or rather with the algebraical example occuring in that *sūtra*, and forty-ninth *sūtra* of the chapter on algebra in the *Brāhma-Siddhānta*" [1]. The *sūtras* in question are in respect of the solution of the quadratic indeterminate equations of the type

$$\sqrt{x + a} = s^2, \quad \sqrt{x - b} = t^2.$$

The *sūtra* in our text[2] is much mutilated, but can be partially restored from the solution. "The sum of the additive and subtractive numbers is divided by an assumed number; the quotient lessened by the same number and halved, is squared and added to the subtractive number" i.e.,

$$x = \left[\frac{1}{2}\left(\frac{a + b}{m} - m\right)\right]^2 + b,$$

Where m is any assumed number. The solution given by Brahmagupta is exactly the same[3].

Again, there is a resemblance between the two works regarding the solution of other type of the quadratic indeterminate equations that is preserved in our text[4] as follows -

$$xy - bx - cy - d = 0$$

The solution obtained is -

$$x = \frac{bc + d}{m} + c, \quad y = b + m;$$

Where m is an assumed number. This solution is closely like the solution found in *Brāhma-sphuṭa-siddhānta* [5], but it differs considerably from the solutions given by

Mahāvīra[6] and Bhāskara[7].

There are also other points of relation between the *Bakhshāli* work and the *Brāhma-sphuṭa-siddhānta*. In Hindu mathematics fractions are usually divided into different classes (*jāti*). One class; which is truely of the most general class consisting of fractions of all the other varieties, is called in the Bakhshāli-work as *pañcamī jāti* (the fifth class)[8]. This is very significant. For according to Śrīdhara[9], Mahāvīra[10], Skandasena and others[11], there are six classes of fractions and the class referred to should be called, according to them, *Bhāga-mātā* (or 'mother fraction'). Bhāskara[12] has reduced the number of classes of fractions to four. It is only Brahmgupta[13] who is known to recognise five classes of fractions. Further we do not find in his work any kind of special technical names, as are commonly found in other Hindu treatises on mathematics. Hence, in the matter of classification of fractions our text is in complete agreement with the work of Brahmgupta. There is an approximate formula in the *Brāhma-sphuṭa-siddhānta* [14], which leads to $(a + x)^2 = a^2 + 2ax + x^2$,

Where x is very small in comparison with a. This can be easily connected with the approximate square-root formula given in our text as -

$$\sqrt{a^2 + 2ax + x^2} = a + x$$

There is also some agreement between our text and *Gaṇita-sāra-saṁgraha*, e.g. the method of reducing fractions to the lowest common denominator is also found in *Gaṇita-Sāra-Saṁgraha*. Again, the name *Kalā savarṇa* for fractions occurs in both the works. There are a few motion problems of the same kind in both the works. The references to religious matters as we find in our text are also met with in the *Gaṇita-Sāra-Saṁgraha* [15].

There are also some points of resemblance between the *Līlāvatī* of Bhāskara and our text. Application of the method of false position for solution of certain

algebraic equations is common to both the works. There is also agreement in the manner of writing groups of fractions. Examples of certain problems are also similar. One problem[16] in our text is proposed to '*Sundrī*' for solution. This naturally reminds us of Bhāskara's '*Līlāvatī*', to whom the problems are addressed for solution in the Bhāskara's work.

The zero symbol has been used in both the works in place of an unknown quantity.

The agreement of our text with the *Triśatika* of Śrīdhara consists in -

i. the manner of writing fractions.

ii. method of writing equations.

iii. the use of the term *rūpa* in connection with the an integer or the integral part of a mixed fraction.

FOREIGN INFLUENCES IN THE BAKHSHĀLI MANUSCRIPT

The village Bakhshali, where our text was discovered, belongs to a region which was the meeting ground of the cross-currents of different cultures, which entered India from the northwest. Amongst these cultures, specially 'noteworthy were the Persian, the Macedonian, the Bactrian-Greek, the Sythian, the Pārthian, the Kuśāna, the Huna, the Arab, etc. Among these cultures which influenced life of the region represented by our Manuscript only the Greeks are credited with possessing history of the study of mathematics. Thus, scholars like Kaye[17] have sought to find traces of Greek influences in the *Bakhshāli Manuscript.*

The method of finding approximate vlaue of surd quantity as found in our text and discussed above has been attributed by Kaye to the Greek Heron (c.200 A.D.). According to him, it occurs in no Indian work earlier than our text. However, it has been convincingly shown by scholars like B.B. Datta[18] that the method was known to ancient Indian mathematicians several centuries before. Thus Āryabhaṭa and Brahmagupta give the formula- $\sqrt{a^2 + r} = a + \frac{r}{2a}$

$$\sqrt[3]{a^3 + r} = a + \frac{r}{3a^2}$$

Rodet holds that a process of approximation to the value of a surd was known to the authors of the *Śulba-sūtras*, the earliest of which was written c.800 B.C.

$$\sqrt{a^2 + r} = a + \frac{r}{2a+1} + \frac{\frac{r}{2a+1}\left(1 - \frac{r}{2a+1}\right)}{2\left(a + \frac{r}{2a+1}\right)} + \varepsilon .$$

where, $\varepsilon =$

$$\left[r - \left\{ \frac{r}{2a+1} + \frac{\frac{r}{(2a+1)}\left(1 - \frac{r}{2a+1}\right)}{2\left(a + \frac{r}{2a+1}\right)} \right\} \left\{ 2a + \frac{r}{2a+1} + \right.\right.$$

$$\left.\left. \frac{\frac{r}{2a+1}\left(1 - \frac{r}{2a+1}\right)}{2\left(a + \frac{r}{2a+1}\right)} \right\} \right] \div 2 \left[a + \frac{r}{2a+1} + \frac{\frac{r}{2a+1}\left(1 - \frac{r}{2a+1}\right)}{2\left(a + \frac{r}{a+1}\right)} \right]$$

This is an approximation of the 4th order.

Putting $a = 1, r = 1$, we get -

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.434},$$

a result well known in *Śulba-sūtras.* [19]

This rule gives an approximation by defect whereas the previous one by excess. Further, this was known to the Greeks, but the second approximation of it was known to the Arabs[20].

Thus Kaye's assertion that 'the square-root rule was not used by the Hindus and was not even noticed by them until the sixteenth century' is not based on facts.

The traces of foreign influence have also been noticed by Kaye[21] in some problems concerning solution of two particular types of linear equations. One set of these problems lead to the simple equations[22] :

$$c - \frac{1}{a_1} c - \frac{1}{a_2}\left(c - \frac{1}{a_1} c\right) - \ldots\ldots\ldots\ldots\ldots = x \quad \ldots (1)$$

$$\text{or } x - \frac{1}{b_1} x - \frac{1}{b_2}\left(x - \frac{1}{b_1} x\right) - \ldots\ldots\ldots\ldots\ldots = x - T. \quad \ldots (2)$$

Equations very similar to (2) appear in the mathematical papyrus of Akhmīm[23]. There is, however, this difference that in the problems of the Bakhshāli work, we are always given what is 'taken away' T from, the original quantity (unknown) by the various specified operations, whereas in the problems of Akhmīm papyrus is given what is 'left' (x - 1) after the operations. Now, the mathematical papyrus of Akhmīm is supposed to have been written between the 6th and 9th centuries. And problems leading to equations similar to (1) and (2) are well known in the Hindu mathematical treatises written in that period, e.g., *Triśatika* [24] (c.750A.D.) and *Gaṇita-sāra-saṁgraha* (850 A.S.)[25]. They are probably contemplated in a rule of *Brāhmasphuṭa-siddānta* (628 A.D.) as is suggested by the illustrative example of the commentator Pṛithudakasvāmi (860 A.D.)[26]. Further there are reasons to believe that the Bakhshāli-work was written long before the period to which the compostiion of the mathe-

work was written long before the period to which the compostiion of the mathematical papyrus of Akhmīm is referred. In such circumstances, observes Datta[27], these problems can not be called to show the stamp of foreign influence.

The simple equations of the type $x_1 + x_2 = a_1$, $x_2 + x_3 = a_2$, $x_3+x_4 = a_3$ occuring in our text are also found in the *Arithmetica* of Diophantus[28]. However, the method of solutions followed in the two works is quite different.

REFERENCES

1. *Indian Antiquary*, vol. XVII, p.37.
2. *The Bakh. Ms.* folio 59 recto, p.215.
3. *Brāhma-sphuṭa-siddhāntà*, xviii, 73, 84.
4. *The Bakh. Ms.* folio 27 recto, p.167.
5. *Brāhma-sphuṭa-siddānta*, xviii, 60.
6. *Gaṇita-sāra-saṁgraha*, vi.284 and vii.112.
7. *Bijagaṇita*, p.123.
8. *The Bakh. Ms.* folio 52 verso, p.165.
9. *Triśatikā*, pp.10-12.
10. *Gaṇita-sāra-saṁgraha*, iii.54.
11. Referred to by Pṛithudakasvāmi (860 A.D.).
    cf. Colebrooke, H.T. *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmgupta and Bhaskara*, London, 1817.
12. *Līlāvatī*, pp.6,7.
13. *Brāhma-sphuṭa-siddhānta*, xii, 8,9.
14. *Ibid.* xii.62.
15. For instance there are mentions of offerings for the purpose of worship *(puja)* to the different jinas in the *Gaṇita-sāra-saṁgraha* (pp.10, 13, 22, 57, 62, 64, 72, etc.) as are in our text, already discussed above, under the title 'The religious-content'. Reference to such religious matters is rarely noticed in any other Hindu work on mathematics.
16. *The Bakh. Ms.* folio 34 recto, p.226.
17. *The Bakh. Ms.* Kaye pp.17, 31, 83.
18. Datta, Bibhutibhushan, "Hindu Contribution to Mathematics", *Bulletin by the*

Mathematical Association, University of Allahabad, Vol.I, (1927-28), p.6.

19. *Ibid.*

20. Smith, *History of mathematics*, Vol. II, p.254.

21. *The Bakh.Ms.* Kaye, p.73.

22. *Ibid.* p.48.

23. Heath, T. *History of Greek Mathematics*, Vol.II, p.544.

24. *Triśatikā*, p.11.

25. *Gaṇita-sāra-saṁgraha*, iii. 127-134; iv.29-32.

26. *Brāhma-sphuṭa-siddhānta*, xii.9 and Pṛithudakasvāmi's commentary thereon; cf. Colebrooke's, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmgupta and Bhaskara*, p.283.

27. Datta, The Bakhshāli Mathematics, *BCMS*, Vol.XXI, p.45.

28. Heath, *History of Greek Mathematics*, Vol.II, p.486.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Source-Book

Kaye, G.R. The Bakhshāli Manuscript, *Archaeological Survey of India, New Imperial Series*, Vol. XLIII, Calcutta, 1927 ; reprinted *Indian* Historical Researches, Vol. 24(i), 24(ii), Cosmo Publications, New Delhi, 1987.

2. Original Sources

Anūyogadvārasūtra, Ed. with Hemacandra's commentary, Jaina Pustakoddhara series no. 37, Bombay 1916.

Āpsatamba Dharma-Sūtra, Ed. and Tr. by U.C. Pandeya (Kashi Sanskrit Series No. 93) Varanasi, 1969.

Āpsatamba-śulba-Sūtra Ed. by D. Srinivasanchar and V.S. Narasimhachar, Mysore Sanskrit series no. 73. Mysore, 1931.

Āryabhaṭīya, Ed. by H. Kern, London, 1874, Ed. by S.K. Pillai, Trivandrum Sanskrit Series, 185, 195.

Aṣṭādhyāyī of Pāṇinī; *Aṣṭādhyāyī-Prakāśika* by Brahmadatta, Delhi, 1955.

Baudhāyana Śulba Sūtra, Ed. by W. Caland, Vol. 3, Calcutta, 1913.

Bhagvati Sūtra, Ed. by N.L. Vaidya with the commentary of Abhayadeva Suri, Srivijayadeva Sursang Series no. 8, 1954.

Brāhma-Sphuṭa-Siddhānta, Ed. with the commentary of Sudhakara Dvivedi, Banaras, 1902, Ed. by R.S. Sharma and Board of editors, 4 vols, Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.

Bṛhajjātaka of Varāhmihira. Ed. with the commentary of Bhaṭṭotpala by Sitaram Jha, Benaras, 1923.

Bṛhaspati Smṛti (reconstructed) by K.V. Rangasvāmi Aiyangar, Baroda, 1941.

Bṛhat-Saṁhitā of Varāhmihira, Ed. by Sudhakara Dvivedi, 2 vols. Benaras, 1895-97; English translation by V. Subrahmanya Sastri and M. Ramakrishna Bhat, 2 Vols., Bangalore, 1947.

Dharma-Sūtras, A study in their origin and development by S.C. Banerjee, Calcutta, 1962.

Digha Nikaya, Ed. by T.W. Rhys Davids and J. Estlin, Pali Text Society, London, 1903.

Divyāvadāna, Ed. by E.B. Cowell and R.A. Neil, Cambridge, 1886.

Gaṇitasārasaṁgraha by Mahāvīra, Ed. with English translationand notes by M. Rangacarya, Madras, 1912.

Gaṇitatilaka of Śrīpati, Ed. by H.R. kapadia, Gaekwad Oriental Series no. 78, Baroda, 1937.

Gautama-Dharma-Sūtra, Ed. by U.C. Pandey, Kashi Sanskrit Series no. 172, Varanasi, 1966.

Gilgit Manuscripts, Ed. by Nalinaksha Dutt, Srinagar, 1939; Second Ed. Delhi, 1984.

Kātyāyana Śulba-Sūtra, Ed. by M.M. Pathaka, Beranas, 1900.

Khaṇḍakhādyaka-Karaṇa, English translation by P.C. Sengupta, University of Calcutta, 1934.

Kṛtyakalpataru, Vyavahāra-Kāṇḍa, Ed. by K.V. Rangaswami Aiyangar.

Līlāvatī of Bhāskara II, Translated into English by H.T. colebrooke vide his '*Algebra with arithmatic and Mensuration from the Sanskrit of Brahmgupta and Bhaskara*, London, 1817; republished by H.C. Banerjee, Allahabad, 1967.

Mahābhāskariya by Bhāskara I, Ed. with his own commentary by Sudhakara Dvivedi, Benaras Sanskrit series no., Benaras, 1910.

Mahābhārata, Ed. Bhandarkar Research Institute Poona.

Mahābhāṣya of Patanjali, Ed. by Bahuvallabha, Asiatic Society of Bengal, Calcutta, 1904.

Mahāmayuri : List of Yaksas, by D.C. Sircar, 1971-72.

Majjihima Nikāya, Ed. by B.J. Kashyap, Nalanda, 1958.

Mānava-Śulba-Sūtra, Ed. with English translation by J.M. Van Gelder, New Delhi, 1961.

Milindapañho, English translation by T.W. Rhys Davids, Oxford, 1890.

Nilamata-Purāṇa, *The Cultural and Literary Study of Nilamata Purana* by Ved Ghai, Srinagar, 1968.

Nirukta of Yāsaka, Ed. by Mukund Jha Bakshi, Bombay, 1930.

Pañcasiddhāntikā of Varāhmihira, Ed. by Thibaut and Sudhakara Dvivedi, 1889; reprint Lahore, 1930.

Pāṭi-Gaṇita of Śrīdhara, Ed. with English translation by K.S. Shukla, Lucknow University, 1959.

Rāmāyaṇa, *Character-Portrayals in the Ramayana of Valmiki*, by Alois Wurm, Delhi, 1976.

Ṛgveda, Ed. by Vishva Bandu, Vishveshvaranand Vedic Research Institute, Hoshiarpur, 1964-65.

Siddhānta-Shekhara, Ed. by Babuaji Misra, Calcutta University, 1936 and 1947.

Śiṣyadhivṛddhida, Ed. by Sudhakara Dvivedi, Benaras, 1886.

Sthānangasūtra, Ed. by N.L. Vaidya, Srivijyadeva Sursangha Series no. 8, 1954.

Sūryaprajñāpati, Ed. by G. Thibaut, JASB, Vol. 49, 1880.

Sūrya Siddhānta, Translated into English by E. Burgess, University of Calcutta, 1935.

Tattvārthādhigamasūtra, Ed. by A.S. Sastri, Govt. Oriental Library Sanskrit series no. 84, Mysore, 1944.

Trilokasāra, Ed. by N. Premi with the commentary of Madhava Candra, Bombay, 1919.

Vedāṅga Jyotiṣa, Ed. by A. Weber, Berlin, 1862; Ed. with English translation by R. Shama Sastri, Mysore, 1936, Ed. by A. Weber, Allahabad, 1960.

Vinaya Pitaka, Ed. by Oldenberg, Tr. by Rhys Davids and H. Oldenberg, SBE XIII, 1881.

South Indian Inscriptions, Vol. IV (in Bengali), Ed. by Rao Bahadur H. Krishna Sastri, Madras, 1923.

Albiruni's India, English tr. by E.C. Sachau, London, IInd Ed., 1910; reprinted, New Delhi, 1983.

Secondary Sources

Aggrawal, V.S., *India as known to Pāninī*, Second Ed. Varanasi, 1969.

Aggrawal, V.S., *vedic Mathematics*, Delhi, 1985.

Bag, A.K., *Mathematics in Ancient and Medieval India*, New Delhi, 1979.

Bamzai, P.N.K., *Socio-Economic History of Kashmir*, New Delhi, 1987.

Bandyopadhyaya, N.C., *Economic Life and Progress in Ancient India*, Allahabad.

Banerjee, J.N., *Development of Hindu Iconography, Calcutta, 1956.*

Bhandarkar, *Lectures on Ancient Indian Numismatics*, Patna, 1921; reprinted Delhi, 1984.

Buhler, G., *Indian Paleography*, English translation by J.F. Fleet, Indian Studies Past and Present, Calcutta, 1959; Reprint, New Delhi, 1980.

Cajori, F., *A History of Elementary Mathematics*, London, 1953.

Cajori, F., *History of Mathematics*, 2nd revised and enlarged edition, New York, 1958.

Cantor, M., *Geschichte der Mathematik*, 1, Part 3, 1907.

Chakraborti, Haripada, *Trade and commerce of Ancient India*, Calcutta.

Chamberlain, C.C., *Numismatics*, London.

Chattopadhyaya, S., *Evolution of Hindu Sects*, New Delhi, 1970.

Colebrooke, H.T., *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara*, London, 1817.

Cunningham, A., *Coins of Medieval India*, London. 1893.

Datta, B.B., *The Science of the Sulba*, A Study in early Hindu Geometry, Calcutta University, 1932.

Datta, B.B. and Singh, A.N., *The History of Hindu Mathematics*, A Source Book, Single Volume edition, Bombay, 1962.

Davids, T.W. Rhys, *Buddhist India*, London, 1903.

Deambi, B.K. Koul, *Corpus of Sarada Inscriptions of Kashmir*, Delhi, 1982.

Deambi, B.K. Koul, *History of Ancient Gandhara and Western Himalayas*, Ariana Publishing House, New Delhi, 1985.

Eves, Howard, *An Introduction to the History of Mathematics*, London, 1964.

Fleet, J.F., *Corpus Inscriptionum Indicarum*, Varanasi, 1963.

Health, T.L., *A History of Greek mathematics*, Oxford, 1921.

Herzfeld, E., *Kusano-Sassanian Coins*, Memoirs of the Archaeological Survey of India, No. 38.

Kak, R.C., *Ancient Monuments of Kashmir*, London, 1933.

Kak, R.C., *Handbook of the Archaeological and Numismatic Sections of the Sri-Pratap Singh Museum*, Srinagar, 1923.

Keith, A.B., *A History of Sanskrit Literature* (Hindi-translation), Delhi, 1967.

Law, B.C., *Historical Geography of Ancient India*, Paris, 1954.

Majumdar, R.C., *The History and Culture of Indian People, The Age of Imperial Unity*, Bhartiya Vidya Bhawan, Bombay.

Ray, S.C., *Early History and Culture of Kashmir*, New Delhi, Second Revised Ed. 1970.

Raychauduri, H.C., *Political History of Ancient India*, Calcutta, 1953.

Sen, S.N., Bag, A.K., Sarma, S.R., *A Bibliography of Sanskrit works on Astronomy and Mathematics*, New Delhi, 1966.

Sen, Sukumar, *Old Persian Inscriptions*, Calcutta 1941.

Shastri, A.M., *India as seen in the Kuṭṭanimata of Damodargupta*, Delhi, 1975.

Sircar, D.C., *Indian Epigraphical Glossary*, Delhi, 1966.

Smith, D.E., *A History of Mathematics*, Dover Publications, 1958.

Smith, *Coins of Ancient India*, Catalogue of coins in the Indian Museum, Varanasi, 1972.

Stein, M.A., *Kalhaṇa's Rājtaraṅgiṇi*, A chronicle of kings of Kashmir, Delhi, 1979.

Vogel,J.Ph., *Antiquities of Chamba State, Part I*, Calcutta, 1911.

Winternitz, M., *A History of Indian Literature*, Delhi, Revised Ed. 1983.

Wurm, Alois, *Character-Portrayal in the Ramayana of Valmiki*, A Systematic Representation, Delhi 1976.

JOURNALS

Archaeological Survey of India, Annual Report, 1913-14.

Bulletin of Calcutta Mathematical Society, Vol. XXI, 1929.

Bulletin of Mathematical Association, University of Allahabad, Vol. I. 1927-28.

Epigraphia Indica, Vols. I, IX, X, XXI, XXII

Indian Antiquary, Vols. XII, XVII; 1883, 1888.

Indian Historical Quarterly, Vol. 5, Reprint, Delhi ,1985.

Journal Asiatique, 1915, Reprint, 1958.

Journal of Asiatic Society of Bengal, Vol. 49, 1880

Journal of Numismatic Society of India, Vol. XV, Varanasi.

DICTIONARIES

A Sanskrit-English Dictionary, Monier Williams.

The Practical Sanskrit-English Dictionary, V.S. Apte.

Encyclopaedia Britannica.

Encyclopaedia of Religion and Ethics, Ed. by James Hastings, New York.

FACSIMILE OF A LEAF OF THE BAKHSHĀLI MANUSCRIPT